CaTT: A type theoretic approach to weak ω-categories

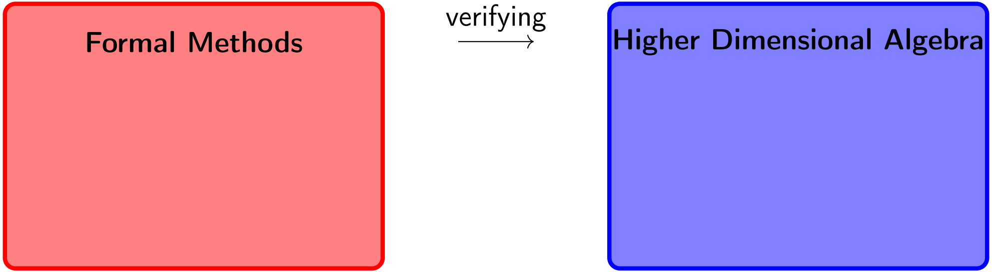
Thibaut Benjamin, Eric Finster, Samuel Mimram

23 February 2021

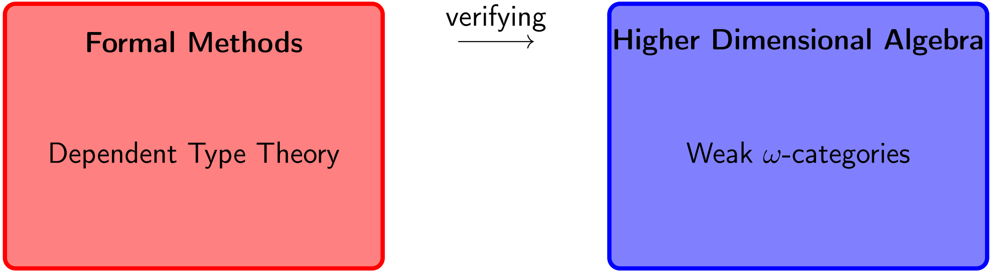
Seminaire LSL

Introduction

Aim



Aim



Formal Methods
Dependent Type Theory
verifying
Higher Dimensional Algebra
Weak ω-categories

Motivations: A bit of category theory

Categories are made of

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Categories are made of

objects

-

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Categories are made of

objects

$\bullet$

arrows

$\bullet \longrightarrow \bullet$

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- A rewriting system is a category : objects = terms, morphisms = rewriting relations.

Motivations: Why categories?

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Provide a common framework to study programming languages, logic and algebra
↳ Semantics

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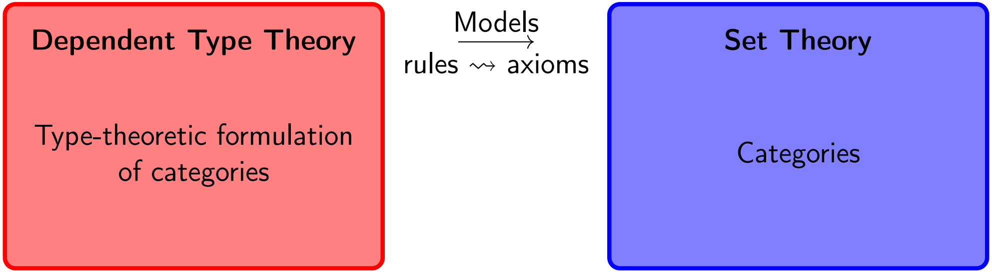
Semantics

Dependent Type Theory

$\xrightarrow{\text{Models}}$

rules $\rightsquigarrow$ axioms

Set Theory



Theorem

The models of the type-theoretic formulation of categories are the categories.

In this presentation

Dependent Type Theory

CaTT

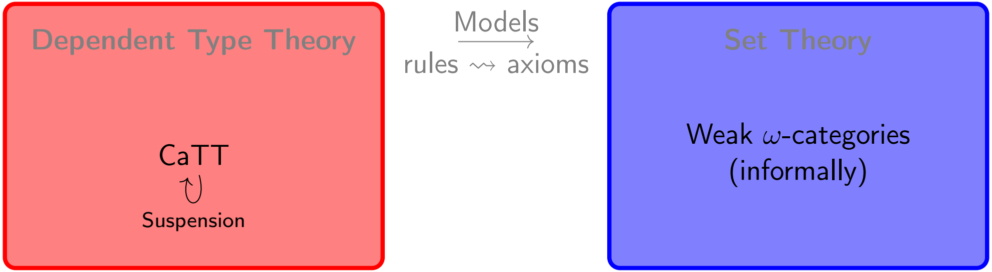
Models $\xrightarrow{\hspace{2em}}$

rules $\rightsquigarrow$ axioms

Set Theory

Weak ω-categories
(informally)

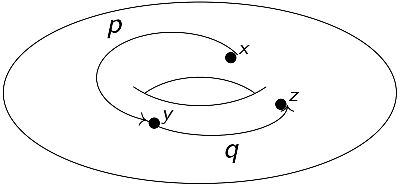
In this presentation



Motivations: When categories are insufficient

Some situation that ought to be categories

- paths in topological spaces



But: The composition is not associative

$$(p * q) * r \neq p * (q * r)$$

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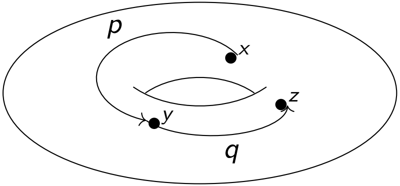

The composition is associative up to homotopy

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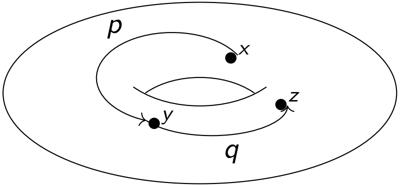

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- Identity types in Martin-Löf type theory

$$\texttt{trans} : \prod A : \mathcal{U},\ \prod x\ y\ z : A,$$
$$x = y \to y = z \to x = z$$
$$\texttt{trans}\ A\ x\ x\ x\ \texttt{refl}\ \texttt{refl}\ := \texttt{refl}$$

But: Transitivity is not associative

$$\texttt{trans}(\texttt{trans}\ p\ q)\ r \not\equiv \texttt{trans}\ p\ (\texttt{trans}\ q\ r)$$

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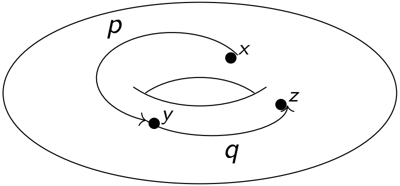

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The composition is associative up to homotopy

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There is a proof term for associativity

$$\texttt{assoc} : \prod A : \mathcal{U},\ \prod\ x\ y\ z\ w : A,\ \prod p : x = y,\ \prod q : y = z,\ \prod r : z = w,\ \texttt{trans}(\texttt{trans}\ p\ q)\ r = \texttt{trans}\ p\ (\texttt{trans}\ q\ r)$$

Higher categories

We can encode this defect into cells of higher dimension

Weak ω-categories

Globular sets

Weak ω-categories = globular sets + compositions

A globular set is composed of

- *points* (or *objects*) : 

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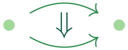
- 2-*cells* : 

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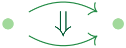
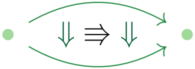
- 2-*cells* : 
- 3-*cells* : 

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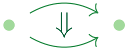
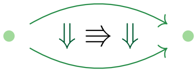
- 2-*cells* : 
- 3-*cells* : 
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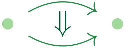
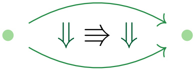
- 2-*cells* : 
- 3-*cells* : 
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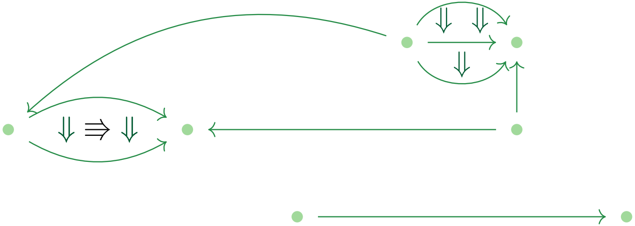

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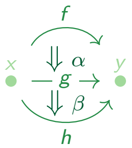

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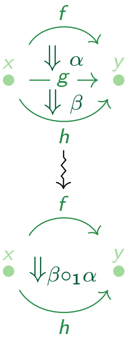

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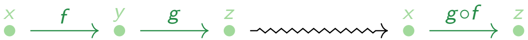

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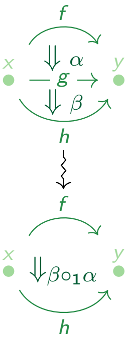

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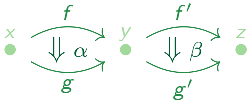

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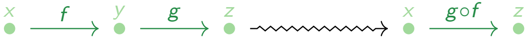

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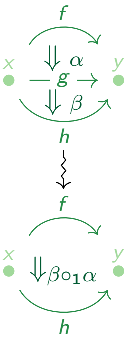

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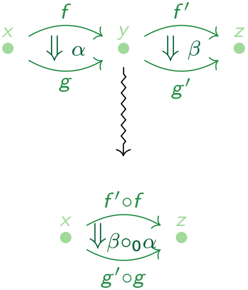

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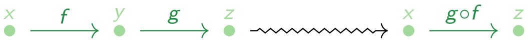

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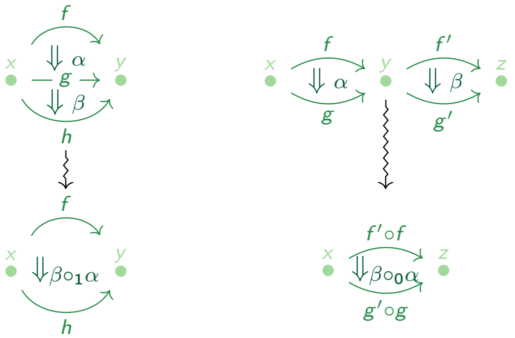

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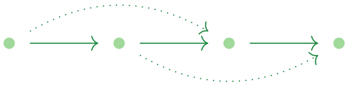
- Arrows: 

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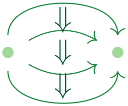

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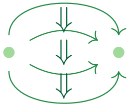

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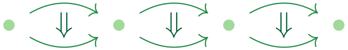

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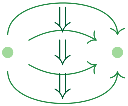

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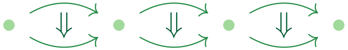
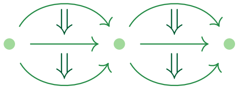


Exchange law



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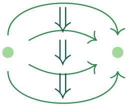
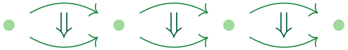
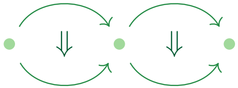


Horizontal composition



Exchange law



Satisfying associativity

The compositions are required to be associative

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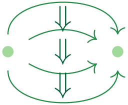

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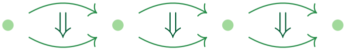

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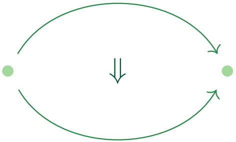

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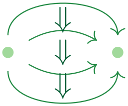

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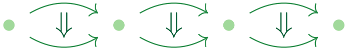

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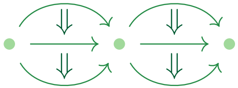

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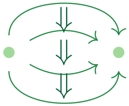

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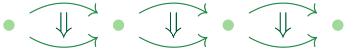

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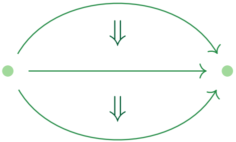

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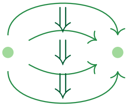

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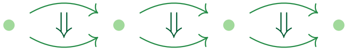

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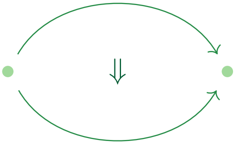

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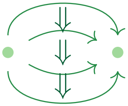

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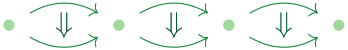

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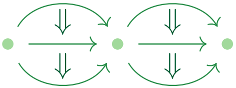

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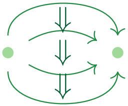

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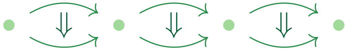

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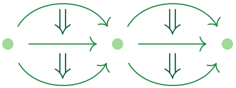

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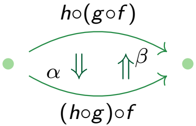

Pasting schemes

The pasting schemes are the globular set that have one unambiguous way of being fully composed

They are well-ordered and do not have any hole

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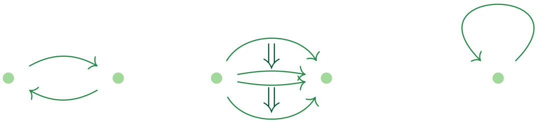

Counter-Examples

The Grothendieck-Maltsiniotis definition

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  Any two ways of composing a same pasting scheme are related by a higher cell

The type theory CaTT

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coh comp (x:*)(y:*)(f:x->y)(z:*)(g:y->z) : x->z
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Globular set

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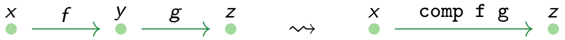

Composition of two arrows

First Examples

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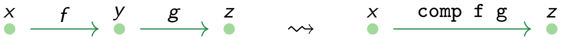

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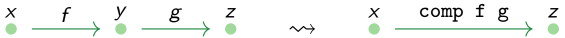

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Identity of an object

Types for cells

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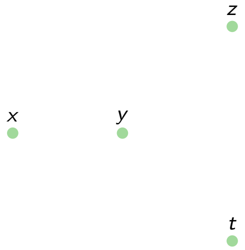
$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : A}{\Gamma \vdash t \underset{A}{\rightarrow} u}$$

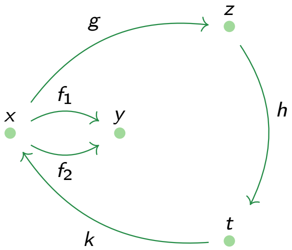
Contexts are diagrams!

$$\Gamma \left\{ \quad x : \star, \qquad y : \star, \qquad z : \star, \qquad t : \star \right.$$



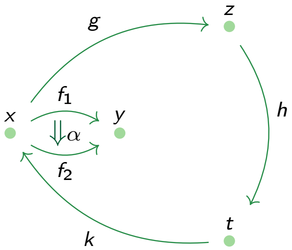
Contexts are diagrams!

$$\Gamma \left\{ \begin{array}{lllll} x : \star, & y : \star, & z : \star, & t : \star & \\ f_1 : x \underset{\star}{\to} y, & f_2 : x \underset{\star}{\to} y, & g : x \underset{\star}{\to} z, & h : z \underset{\star}{\to} t, & k : t \underset{\star}{\to} x \\ & & & & \\ & & & & \end{array} \right.$$



Contexts are diagrams!

$$\Gamma \begin{cases} x : \star, \quad y : \star, \quad z : \star, \quad t : \star \\ f_1 : x \xrightarrow[\star]{} y, \quad f_2 : x \xrightarrow[\star]{} y, \quad g : x \xrightarrow[\star]{} z, \quad h : z \xrightarrow[\star]{} t, \quad k : t \xrightarrow[\star]{} x \\ \alpha : f_1 \xrightarrow[x \xrightarrow[\star]{} y]{} f_2 \end{cases}$$



PS-contexts

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- This judgment is decidable with an algorithm

$$\frac{}{x : \star \vdash_{\mathsf{ps}} x : \star} \qquad \frac{\Gamma \vdash_{\mathsf{ps}} x : A}{\Gamma, y : A, f : x \underset{A}{\longrightarrow} y \vdash_{\mathsf{ps}} f : x \underset{A}{\longrightarrow} y}$$

$$\frac{\Gamma \vdash_{\mathsf{ps}} f : x \underset{A}{\longrightarrow} y}{\Gamma \vdash_{\mathsf{ps}} y : A} \qquad \frac{\Gamma \vdash_{\mathsf{ps}} x : \star}{\Gamma \vdash_{\mathsf{ps}}}$$

Source and target

Every ps-context Γ has a source $\partial^{-}(\Gamma)$ and a target $\partial^{+}(\Gamma)$ (which are ps-contexts themselves)

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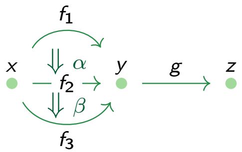

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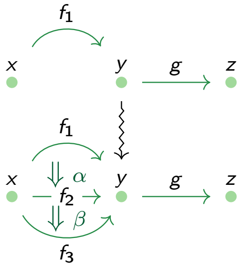

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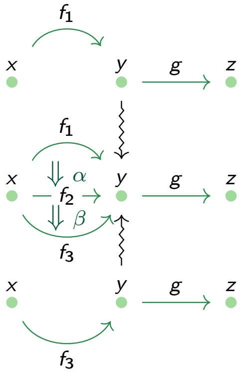

Example

Interpretation

- A term $\Gamma \vdash t : A$ corresponds to a composition of certain cells of Γ.

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- Case of ps-contexts $\Gamma \vdash_{\mathsf{ps}}$:

$$\left.\begin{array}{l}\Gamma \vdash t : A \\ \mathrm{Var}(t : A) = \mathrm{Var}(\Gamma)\end{array}\right\}$$

t is a way of composing completely the ps-context Γ.

Operations and coherences

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$$\Gamma = x : \star, y : \star, f : x \underset{\star}{\to} y, z : \star, g : y \underset{\star}{\to} z$$
$$\Gamma \vdash \mathtt{comp} : x \to z$$

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$$\Gamma \vdash \texttt{assoc} : \texttt{comp f (comp g h)} \to \texttt{comp (comp f g) h}$$

Example

Suspension in CaTT

A problem with CaTT

Writing in CaTT may be very tedious :

$$x \rightsquigarrow x \xrightarrow{\mathrm{id}_x} x$$

```
coh id (x:*):x->x
```

$$x \xrightarrow{f} y \quad \rightsquigarrow \quad x \underset{f}{\overset{f}{\rightrightarrows}} y \quad (\Downarrow \mathrm{id}_f)$$

```
coh id2 (x:*)(y:*)(f:x->y):f->f
```

A problem with CaTT

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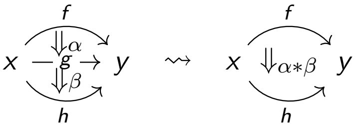
$$x \xrightarrow{f} y \xrightarrow{g} z \quad \rightsquigarrow \quad x \xrightarrow{g \circ f} z$$

```
coh comp (x:*)(y:*)(f:x->y)
     (z:*)(g:y->z):x->z
```




```
coh vcomp (x:*)(y:*)(f:x->y)(g:x->y)
     (a:f->g)(h:x->y)(b:g->h):f->h
```

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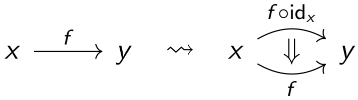
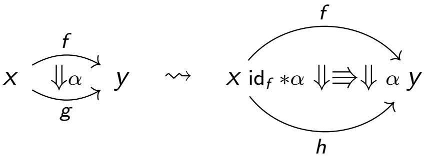



```
coh unitl (x:*)(y:*)(f:x->y)
      :comp (id x) f -> f
```




```
coh unitl2 (x:*)(y:*)(f:x->y)(g:x->y)
      (a:f->g):vcomp (id f) a -> a
```

A solution: the suspension

Idea: Write only the left term and let the software generate the right one.

```
coh id (x:*):x->x
```

suspension $\rightsquigarrow$

```
coh id2 (x:*)(y:*)(f:x->y):f->f
```

Practice: Generate the terms on the fly using the dimension of the argument.

```
coh id (x:*):x->x
let ex (x:*)(y:*)(f:x->y)=id f
```

$\rightsquigarrow$

```
coh id (x:*):x->x
coh id2 (x:*)(y:*)(f:x->y):f->f
let ex (x:*)(y:*)(f:x->y)=id2 f
```

Definition of the suspension

Induction over the syntax: Define the suspension as a meta-operation on the syntax of the type theory.

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$$\Sigma x = x \qquad \Sigma(\mathrm{coh}_{\Gamma,A}[\gamma]) = \mathrm{coh}_{\Sigma\Gamma,\Sigma A}[\Sigma\gamma]$$

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Induction over the syntax: Define the suspension as a meta-operation on the syntax of the type theory.

$$\Sigma\varnothing = (x_0 : \star, y_0 : \star) \qquad \Sigma(\Gamma, x : A) = \Sigma\Gamma, x : \Sigma A$$

$$\Sigma\star = x_0 \underset{\star}{\rightarrow} y_0 \qquad \Sigma(t \underset{A}{\rightarrow} u) = \Sigma t \underset{\Sigma A}{\longrightarrow} \Sigma u$$

$$\Sigma x = x \qquad \Sigma(\mathrm{coh}_{\Gamma,A}[\gamma]) = \mathrm{coh}_{\Sigma\Gamma,\Sigma A}[\Sigma\gamma]$$

$$\Sigma(\langle\rangle) = \langle x_0 \mapsto x_0, y_0 \mapsto y_0\rangle \qquad \Sigma(\langle\gamma, x \mapsto t\rangle) = \langle\Sigma\gamma, x \mapsto \Sigma t\rangle$$

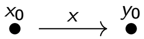
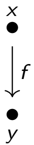
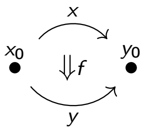
Illustration

On contexts:

Γ	$\Sigma\Gamma$
$\overset{x}{\bullet}$	$\overset{x_0}{\bullet} \xrightarrow{x} \overset{y_0}{\bullet}$
$(x : \star)$	$(x_0 : \star, y_0 : \star, x : x_0 \underset{\star}{\rightarrow} y_0)$

Illustration

On contexts:

Γ	$\Sigma\Gamma$
	
$(x : \star)$	$(x_0 : \star, y_0 : \star, x : x_0 \underset{\star}{\to} y_0)$
	
$(x : \star, y : \star, f : x \underset{\star}{\to} y)$	$(x_0 : \star, y_0 : \star, x : x_0 \to y_0, y : x_0 \to y_0, f : x \to y)$

Well-definedness of the suspension

$\Sigma\varnothing = (x_0 : *, y_0 : *)$	$\Sigma(\Gamma, x : A) = \Sigma\Gamma, x : \Sigma A$
$\Sigma\star = x_0 \to y_0$	$\Sigma(t \underset{A}{\to} u) = \Sigma t \underset{\Sigma A}{\longrightarrow} \Sigma u$
$\Sigma x = x$	$\Sigma(\mathrm{coh}_{\Gamma,A}[\gamma]) = \mathrm{coh}_{\Sigma\Gamma,\Sigma A}[\Sigma\gamma]$
$\Sigma(\langle\rangle) = \langle x_0 \mapsto x_0, y_0 \mapsto y_0\rangle$	$\Sigma(\langle\gamma, x \mapsto t\rangle) = \langle\Sigma\gamma, x \mapsto \Sigma t\rangle$

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Theorem (B., Mimram)

The following rules are derivable

$$\frac{\Gamma \vdash}{\Sigma\Gamma \vdash} \qquad \frac{\Gamma \vdash A}{\Sigma\Gamma \vdash \Sigma A} \qquad \frac{\Gamma \vdash t : A}{\Sigma\Gamma \vdash \Sigma t : \Sigma A} \qquad \frac{\Delta \vdash \gamma : \Gamma}{\Sigma\Delta \vdash \Sigma\gamma : \Sigma\Gamma}$$

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Proof.

By induction over the rules of the theory

Example: A development using the suspension

```
coh id (x:*):x->x
coh comp (x:*)(y:*)(f:x->y)(z:*)(g:y->z):x->z
coh unit (x:*)(y:*)(f:x->y):comp (id x) f -> f
let unit2 (x:*)(y:*)(f:x->y)(g:x->y)(a:f->g)=unit a
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Internally:

- The system computes the difference in the dimension of the argument
  `unit` expects an argument of dimension 1,a is of dimension 2 $\rightsquigarrow$ suspend once

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- Compute the suspension of `unit`
- Compute the suspension of `comp`

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coh comp (x:*)(y:*)(f:x->y)(z:*)(g:y->z):x->z
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let unit2 (x:*)(y:*)(f:x->y)(g:x->y)(a:f->g)=unit a
```

Internally:

- The system computes the difference in the dimension of the argument
  `unit` expects an argument of dimension 1,a is of dimension 2 $\rightsquigarrow$ suspend once
- Compute the suspension of `unit`
- Compute the suspension of `comp`
- compute the suspension of `id`

Thank you!

Try it yourself

`https://thibautbenjamin.github.io/catt/`