CaTT: A type theoretic approach to weak ω-categories

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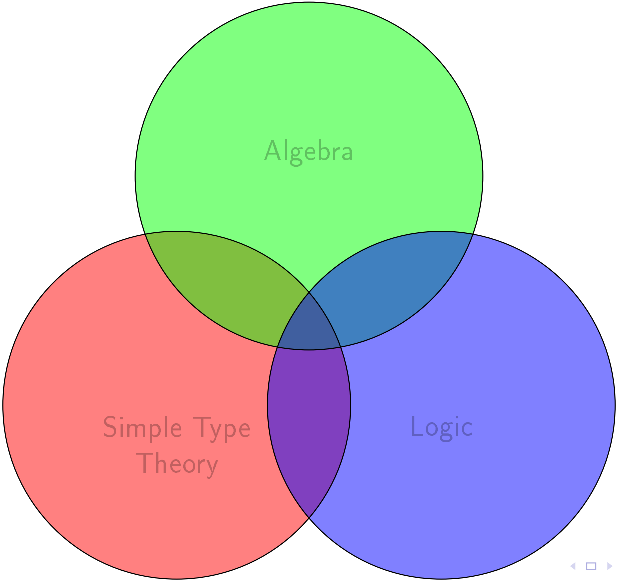
15 February 2021

Seminaire LACL

Introduction

The panorama



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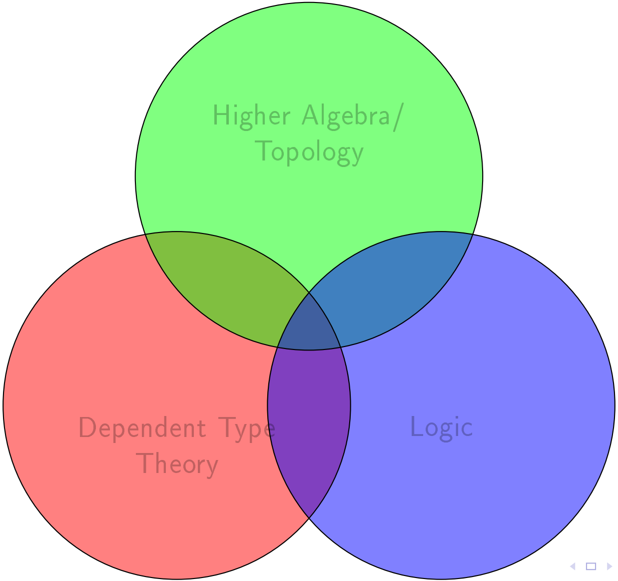

Motivations: A bit of category theory

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-

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- A rewriting system is a category : objects = terms, morphisms = rewriting relations.

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Provide a common framework to study programming languages, logic and algebra
↳ Semantics

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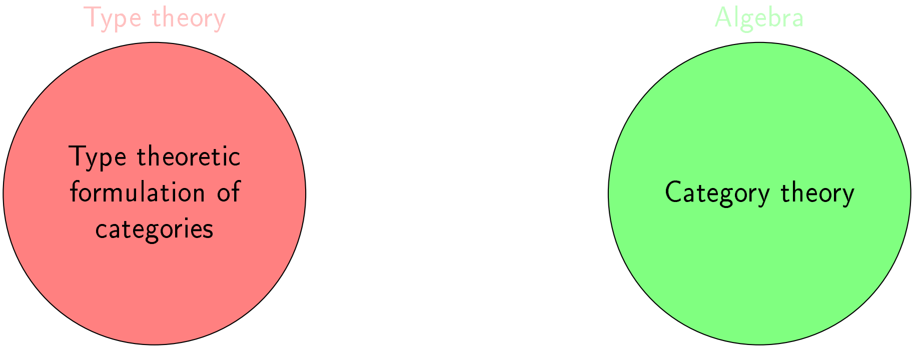
Semantics


Type theory
Type theoretic formulation of categories
Algebra
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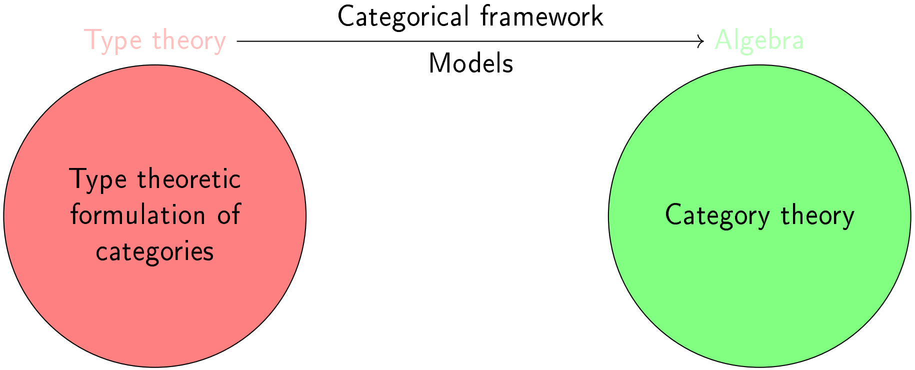
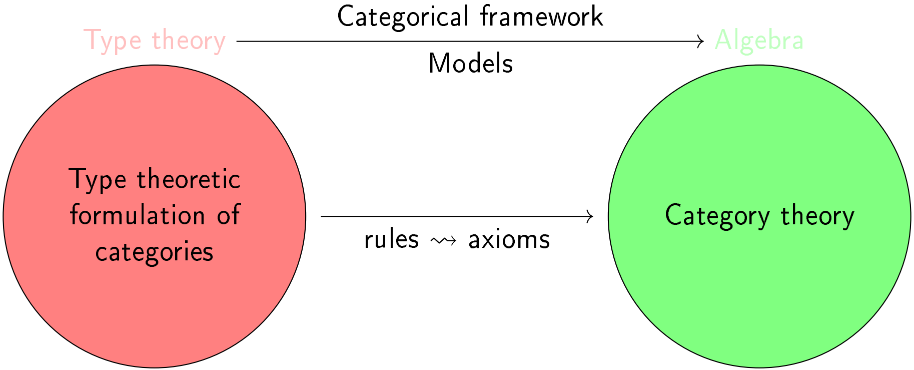

Type theory
Categorical framework
Models
Algebra
Type theoretic formulation of categories
Category theory

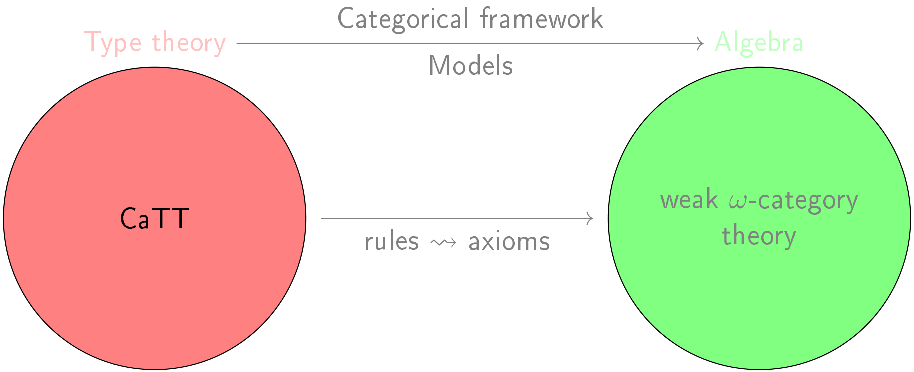
Semantics



Theorem

The models of the type-theoretic formulation of categories are the categories.

In this presentation



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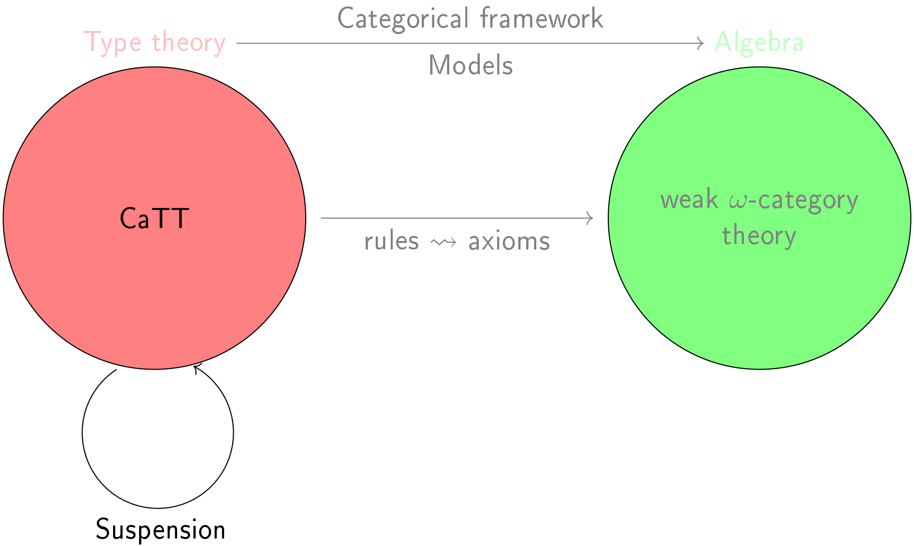
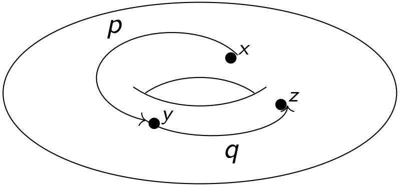


Motivations: When categories are insufficient

Some situation that ought to be categories

- paths in topological spaces



But: The composition is not associative

$$(p * q) * r \neq p * (q * r)$$

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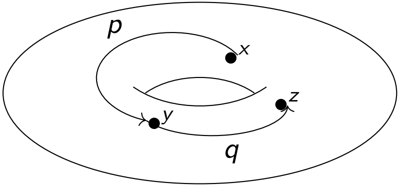

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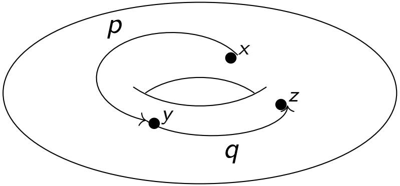

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- Identity types in Martin-Löf type theory

$$\texttt{trans} : \prod A : \mathcal{U}, \prod x\ y\ z : A, \\ x = y \to y = z \to x = z$$

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But: Transitivity is not associative

$$\texttt{trans}(\texttt{trans}\ p\ q)\ r \not\equiv \texttt{trans}\ p\ (\texttt{trans}\ q\ r)$$

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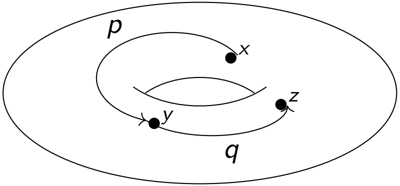

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There is a proof term for associativity

$$\texttt{assoc} : \prod A : \mathcal{U},\ \prod\ x\ y\ z\ w : A,$$
$$\prod p : x = y,\ \prod q : y = z,\ \prod r : z = w,$$
$$\texttt{trans}(\texttt{trans}\ p\ q)\ r = \texttt{trans}\ p\ (\texttt{trans}\ q\ r)$$

We can encode this defect into cells of higher dimension

Weak ω-categories

Globular sets

Weak ω-categories = globular sets + compositions

Globular sets

A globular set is composed of

- *points* (or *objects*) : •

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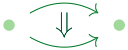
- 2-*cells* : 

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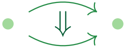
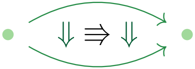
- 2-*cells* : 
- 3-*cells* : 

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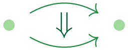
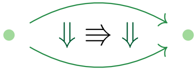
- 2-*cells* : 
- 3-*cells* : 
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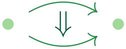
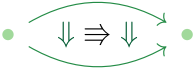
- 2-*cells* : 
- 3-*cells* : 
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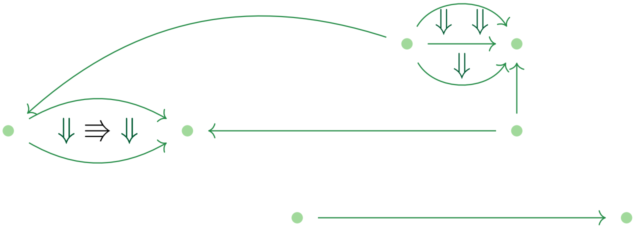

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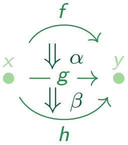

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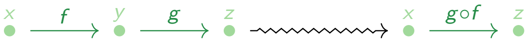

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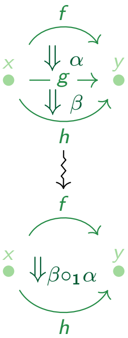

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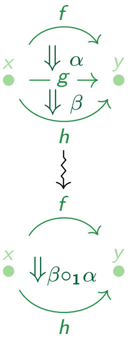

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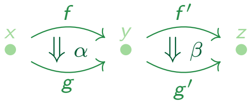

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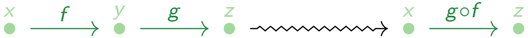

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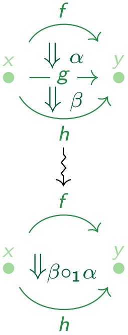

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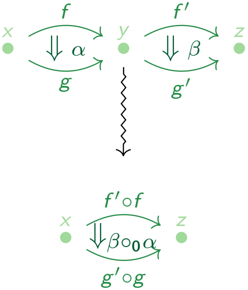

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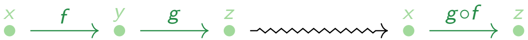

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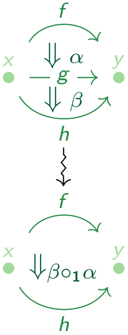

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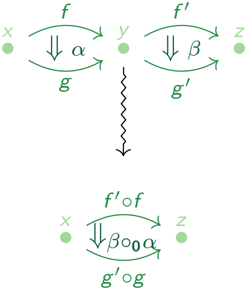

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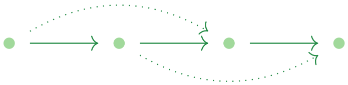
- Arrows: 

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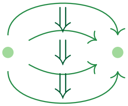

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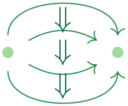

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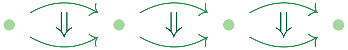

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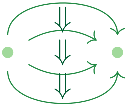
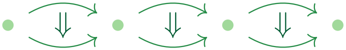
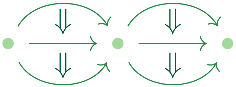


Horizontal composition



Exchange law



Satisfying associativity

The compositions are required to be associative

- Arrows:

- 2-cells:

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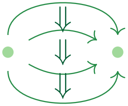

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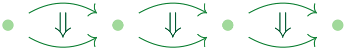

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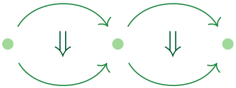

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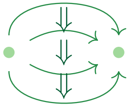

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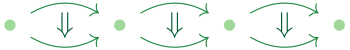

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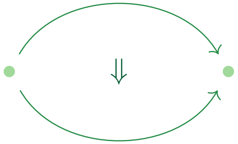

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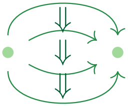

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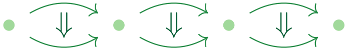

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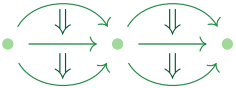

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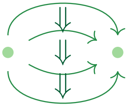

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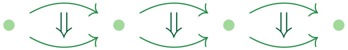

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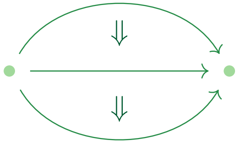

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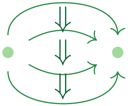

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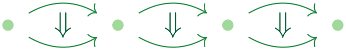

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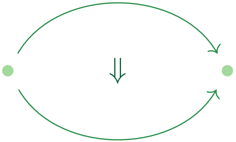

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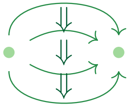

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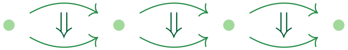

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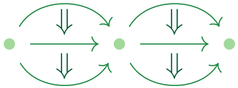

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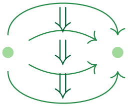

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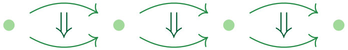

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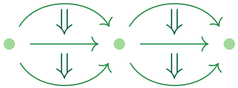

- etc

Everything is weak!

All the above-mentioned equalities are weak: they are equivalences

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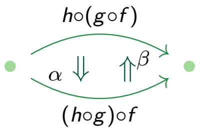

Pasting schemes

The pasting schemes are the globular set that have one unambiguous way of being fully composed

They are well-ordered and do not have any hole

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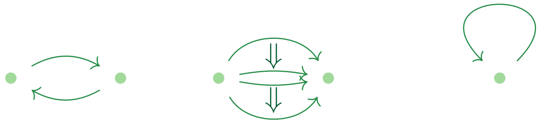

Counter-Examples

The Grothendieck-Maltsiniotis definition

- Existence of compositions:
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The Grothendieck-Maltsiniotis definition

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The type theory CaTT

First Examples

```
coh comp (x:*)(y:*)(f:x->y)(z:*)(g:y->z): x->z
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Globular set

$$x \xrightarrow{f} y \xrightarrow{g} z$$

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Return type

Globular set

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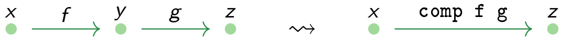

Composition of two arrows

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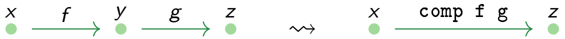

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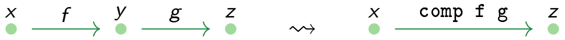

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Identity of an object

Types for cells

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Types for cells

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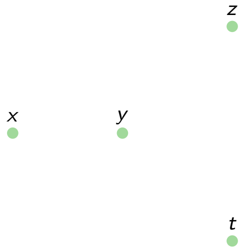
- a type $\to$ for all $\geq$ 1-cells

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : A}{\Gamma \vdash t \underset{A}{\to} u}$$

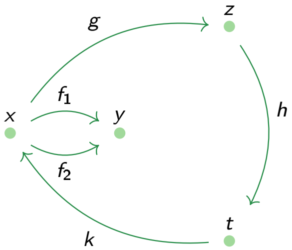
Contexts are diagrams!

$$\Gamma \left\{ x : \star, \quad y : \star, \quad z : \star, \quad t : \star \right.$$



Contexts are diagrams!

$$\Gamma \begin{cases} x : \star, \quad y : \star, \quad z : \star, \quad t : \star \\ f_1 : x \underset{\star}{\rightarrow} y, \quad f_2 : x \underset{\star}{\rightarrow} y, \quad g : x \underset{\star}{\rightarrow} z, \quad h : z \underset{\star}{\rightarrow} t, \quad k : t \underset{\star}{\rightarrow} x \end{cases}$$



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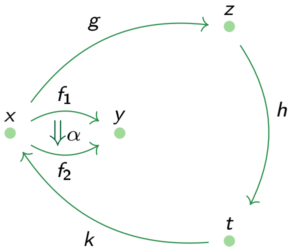

PS-contexts

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They are recognized by a judgment $\Gamma \vdash_{\mathsf{ps}}$

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- This judgment is decidable with an algorithm

$$\frac{}{x : \star \vdash_{\mathsf{ps}} x : \star} \qquad \frac{\Gamma \vdash_{\mathsf{ps}} x : A}{\Gamma, y : A, f : x \underset{A}{\longrightarrow} y \vdash_{\mathsf{ps}} f : x \underset{A}{\longrightarrow} y}$$

$$\frac{\Gamma \vdash_{\mathsf{ps}} f : x \underset{A}{\longrightarrow} y}{\Gamma \vdash_{\mathsf{ps}} y : A} \qquad \frac{\Gamma \vdash_{\mathsf{ps}} x : \star}{\Gamma \vdash_{\mathsf{ps}}}$$

Source and target

Every ps-context Γ has a source $\partial^-(\Gamma)$ and a target $\partial^+(\Gamma)$ (which are ps-contexts themselves)

Intuitively, composing fully the ps-context yields a cell from its source to its target

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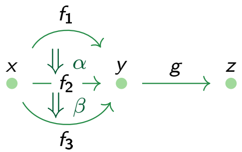

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$\partial^-(\Gamma)$

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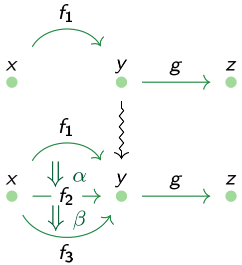

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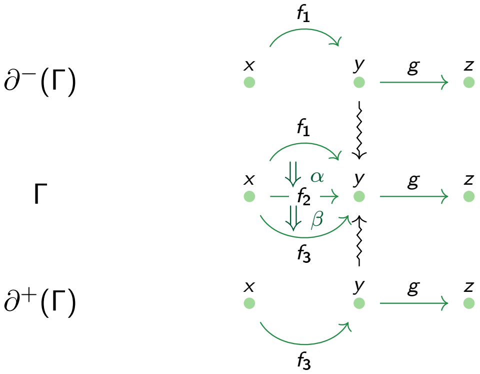

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Interpretation

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- Case of ps-contexts $\Gamma \vdash_{\mathrm{ps}}$:

$$\left.\begin{array}{l}\Gamma \vdash t : A \\ \mathrm{Var}(t : A) = \mathrm{Var}(\Gamma)\end{array}\right\}$$

  t is a way of composing completely the ps-context Γ.

Operations and coherences

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Rule for generating these compositions:

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$$\begin{aligned} \mathrm{Var}(t : A) &= \mathrm{Var}(\partial^{-}(\Gamma)) \\ \mathrm{Var}(u : A) &= \mathrm{Var}(\partial^{+}(\Gamma)) \end{aligned}$$

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$$\Gamma = x : \star, y : \star, f : x \underset{\star}{\to} y, z : \star, g : y \underset{\star}{\to} z$$
$$\Gamma \vdash \mathtt{comp} : x \to z$$

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$$\Gamma = x : \star, y : \star, f : x \underset{\star}{\to} y, z : \star, g : y \underset{\star}{\to} z, w : \star, h : z \underset{\star}{\to} w$$

$$\Gamma \vdash \mathsf{assoc} : \texttt{comp f (comp g h)} \to \texttt{comp (comp f g) h}$$

Example

Suspension in CaTT

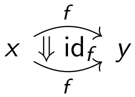
A problem with CaTT

Writing in CaTT may be very tedious :

$$x \rightsquigarrow x \xrightarrow{\mathrm{id}_x} x$$

```
coh id (x:*):x->x
```

$$x \xrightarrow{f} y \rightsquigarrow$$



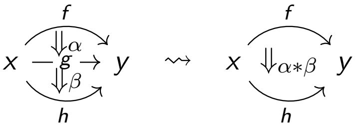

```
coh id2 (x:*)(y:*)(f:x->y):f->f
```

A problem with CaTT

Writing in CaTT may be very tedious :

$$x \xrightarrow{f} y \xrightarrow{g} z \quad \rightsquigarrow \quad x \xrightarrow{g \circ f} z$$

```
coh comp (x:*)(y:*)(f:x->y)
    (z:*)(g:y->z):x->z
```




```
coh vcomp (x:*)(y:*)(f:x->y)(g:x->y)
    (a:f->g)(h:x->y)(b:g->h):f->h
```

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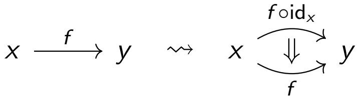
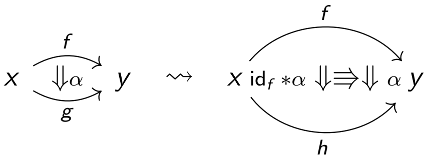



```
coh unitl (x:*)(y:*)(f:x->y)
    :comp (id x) f -> f
```




```
coh unitl2 (x:*)(y:*)(f:x->y)(g:x->y)
    (a:f->g):vcomp (id f) a -> a
```

A solution: the suspension

Idea: Write only the left term and let the software generate the right one.

```
coh id (x:*):x->x
```

suspension $\rightsquigarrow$

```
coh id2 (x:*)(y:*)(f:x->y):f->f
```

Practice: Generate the terms on the fly using the dimension of the argument.

```
coh id (x:*):x->x
let ex (x:*)(y:*)(f:x->y)=id f
```

$\rightsquigarrow$

```
coh id (x:*):x->x
coh id2 (x:*)(y:*)(f:x->y):f->f
let ex (x:*)(y:*)(f:x->y)=id2 f
```

Definition of the suspension

Induction over the syntax: Define the suspension as a meta-operation on the syntax of the type theory.

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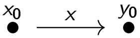
$$\Sigma(\langle\rangle) = \langle x_0 \mapsto x_0, y_0 \mapsto y_0\rangle \qquad \Sigma(\langle\gamma, x \mapsto t\rangle) = \langle\Sigma\gamma, x \mapsto \Sigma t\rangle$$

Illustration

On contexts:

Γ	$\Sigma\Gamma$
	
$(x : \star)$	$(x_0 : \star, y_0 : \star, x : x_0 \underset{\star}{\rightarrow} y_0)$

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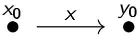
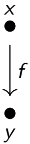
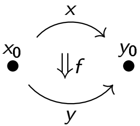
 $(x : \star)$	 $(x_0 : \star, y_0 : \star, x : x_0 \underset{\star}{\rightarrow} y_0)$
 $(x : \star, y : \star, f : x \underset{\star}{\rightarrow} y)$	 $(x_0 : \star, y_0 : \star, x : x_0 \rightarrow y_0, y : x_0 \rightarrow y_0, f : x \rightarrow y)$

Well-definedness of the suspension

$\Sigma\varnothing = (x_0 : *, y_0 : *)$	$\Sigma(\Gamma, x : A) = \Sigma\Gamma, x : \Sigma A$
$\Sigma\star = x_0 \to y_0$	$\Sigma(t \underset{A}{\to} u) = \Sigma t \underset{\Sigma A}{\longrightarrow} \Sigma u$
$\Sigma x = x$	$\Sigma(\mathrm{coh}_{\Gamma,A}[\gamma]) = \mathrm{coh}_{\Sigma\Gamma,\Sigma A}[\Sigma\gamma]$
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Theorem (B., Mimram)

The following rules are derivable

$$\frac{\Gamma \vdash}{\Sigma\Gamma \vdash} \qquad \frac{\Gamma \vdash A}{\Sigma\Gamma \vdash \Sigma A} \qquad \frac{\Gamma \vdash t : A}{\Sigma\Gamma \vdash \Sigma t : \Sigma A} \qquad \frac{\Delta \vdash \gamma : \Gamma}{\Sigma\Delta \vdash \Sigma\gamma : \Sigma\Gamma}$$

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$$

Proof.

By induction over the rules of the theory

□

Example: A development using the suspension

```
coh id (x:*):x->x
coh comp (x:*)(y:*)(f:x->y)(z:*)(g:y->z):x->z
coh unit (x:*)(y:*)(f:x->y):comp (id x) f -> f
let unit2 (x:*)(y:*)(f:x->y)(g:x->y)(a:f->g)=unit a
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- The system computes the difference in the dimension of the argument
  `unit` expects an argument of dimension 1,a is of dimension 2 $\rightsquigarrow$ suspend once

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- Compute the suspension of `unit`
- Compute the suspension of `comp`
- compute the suspension of `id`

A categorical view on the suspension

- One can define the syntactic category $\mathcal{S}_{\mathsf{CaTT}}$
  - Objects = contexts
  - Morphisms = substitutions

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Problem: This operation is not always well-defined. It only defines a partial endomorphism of CwFs

Thank you!