Globular weak ω-categories as models of a type theory

Thibaut Benjamin, Eric Finster, Samuel Mimram

CEA, LIST

HoTT/UF, 17/07/21

A short introduction of CaTT

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  A type theory that describes ω-groupoids = the combinatorics of the identity type in *HoTT*

- It aims at proving *coherences*.
  It cannot define an ω-categories, but can only prove their combinatorics

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Grothendieck weak ω-groupoids [4] — directionality ⟶ Grothendieck-Maltsiniotis weak ω-cateogries [5]

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Brunerie's type theory [2] —— directionality ⟶ CaTT [3]

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- For today: Give a few examples with CaTT and study the semantics for globular sets.

A few examples in CaTT

Example

https://thibautbenjamin.github.io/catt/

```
coh comp (x:*)(y:*)(f:x->y)(z:*)(g:y->z) : x->z
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diagram

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- `coh` — keyword: coherence declaration
- `comp` — name of the coherence `comp`
- `(x:*)(y:*)(f:x->y)(z:*)(g:y->z)` — diagram

- `x->z` — result type

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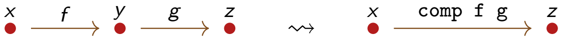

morphism composition

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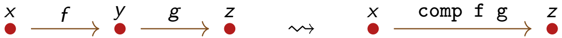

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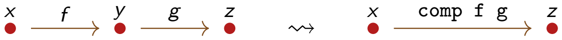

morphism composition

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identity of objects

Some more advanced examples

```
coh unit (x:*)(y:*)(f:x->y) : comp (id x) f -> f
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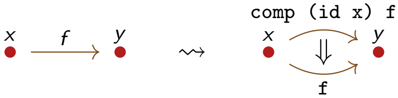

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```
coh vcomp
    (x:*)(y:*)(f:x->y)(g:x->y)(a:f->g)(h:x->y)(b:g->h) :
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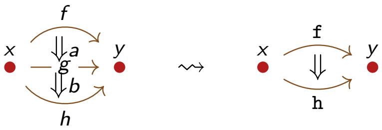

Some more advanced examples

```
coh whiskr
    (x:*)(y:*)(f:x->y)(f':x->y)(a:f->f')(z:*)(g:y->z) :
    comp f g -> comp f' g
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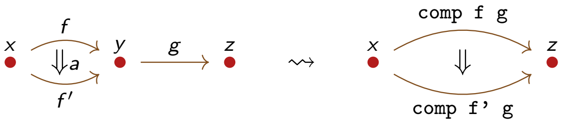

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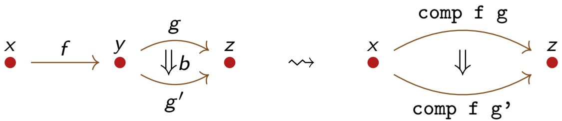

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coh exch
    (x:*)(y:*)(f:x->y)(f':x->y)(a:f->f')
         (z:*)(g:y->z)(g':y->z)(b:g->g') :
    vcomp (whiskr a g) (whiskl f' b) ->
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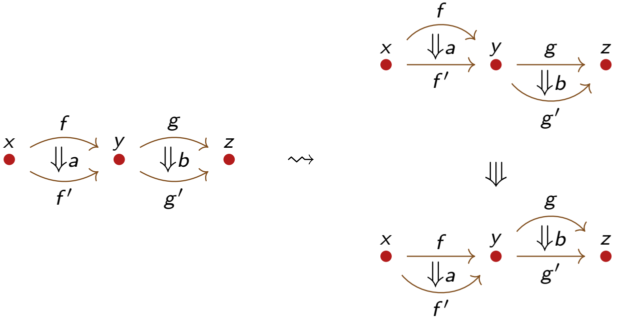

The theory GSeTT and its models

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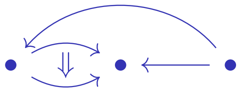

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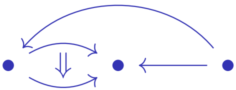

- They are presheaves over the base category

$$\mathcal{G} = \quad 0 \underset{t}{\overset{s}{\rightrightarrows}} 1 \underset{t}{\overset{s}{\rightrightarrows}} 2 \underset{t}{\overset{s}{\rightrightarrows}} \cdots \qquad \begin{array}{c} ts = ss \\ st = tt \end{array}$$

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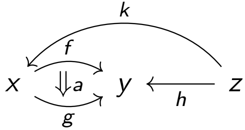
- We can use it to describe the shapes of cells

The syntactic category

- A context Γ induces a finite globular set $V\Gamma$
  We can read off directly a globular set from a context (like before)

```
(x:*,y:*,z:*,
f:x->y,g:x->y,
h:z->y,k:z->x,
a:f->g)
```

$\rightsquigarrow$



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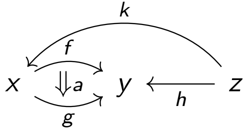

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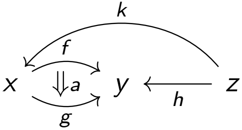

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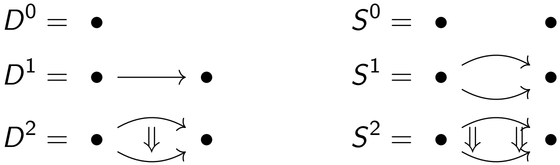
Theorem
The syntactic category $\mathcal{S}_{GSeTT}$ (objects=contexts, morphisms=substitutions) is the opposite of the category of finite globular sets FinGSet.

Disks and sphere contexts

Some special contexts in the theory GSeTT.



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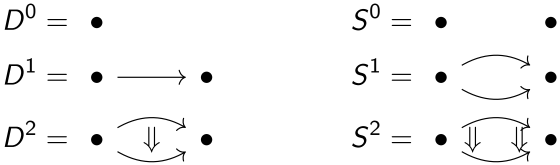

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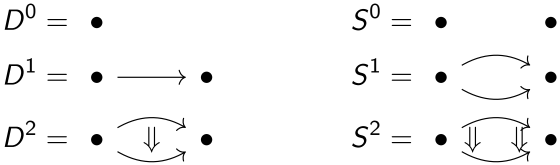

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This gives a functor $D^{\bullet} : \mathcal{G}^{\mathrm{op}} \to \mathcal{S}_{\mathsf{GSeTT}}$

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The models of the theory GSeTT are equivalent to globular sets.

Idea of the proof

Models = CwF morphisms $\mathcal{S}_{\mathsf{GSeTT}} \to \mathsf{Set}$.

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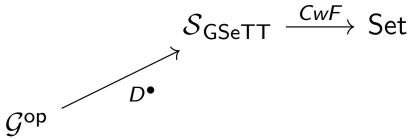

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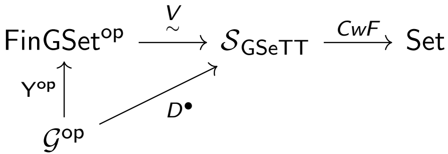

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$$\begin{aligned} \mathsf{Mod}(\mathcal{S}_{\mathsf{GSeTT}}) &\simeq [\mathsf{FinGSet}^{\mathsf{op}}, Set]_{\text{finite lim}} \\ &\simeq [\mathcal{G}^{\mathsf{op}}, Set] \end{aligned}$$

□

Semantics of the theory GSeTT

The universal property of disks

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$$\mathrm{Tm}(\Gamma) \simeq \bigsqcup_{n \geq 0} \mathcal{S}_{\mathsf{GSeTT}}(\Gamma, D^n)$$

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- V is the nerve functor associated to the functor $D^\bullet$
- There is a natural isomorphism

$$\mathcal{S}_{\mathrm{GSeTT}}(\Delta, \Gamma) \simeq \mathrm{FinGSet}(V\Gamma, V\Delta) \tag{1}$$

Models of the theory (2)

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A CwF morphism $\mathcal{S}_{\mathsf{GSeTT}} \to \mathcal{C}$ induces a globular CwF on $\mathcal{C}$

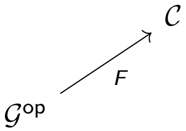
The initiality theorem

Theorem

The category $\mathcal{S}_{GSeTT}$ is the initial globular CwF

Proof.

Consider a globular CwF



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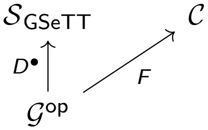

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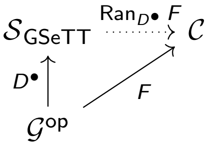

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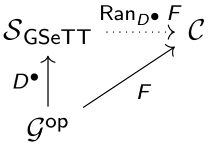

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Corollary

The models of $\mathcal{S}_{GSeTT}$ are equivalent to globular CwF structures on Set.

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globular CwF structures on Set are equivalent to functors $\mathcal{G}^{\mathsf{op}} \to \mathrm{Set}$, i.e., globular sets. □

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This proof is the same as the previous one in essence, but

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A word on the semantics of CaTT

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- ▷ Cat-CwF structures on Set are equivalent to weak ω-categories

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