

Hom ω -categories of a computad are free

Thibaut Benjamin* Ioannis Markakis†

November 14, 2024

Abstract

We provide a new description of the hom functor on weak ω -categories, and we show that it admits a left adjoint that we call the suspension functor. We then show that the hom functor preserves the property of being free on a computad, in contrast to the hom functor for strict ω -categories. Using the same technique, we define the opposite of an ω -category with respect to a set of dimensions, and we show that this construction also preserves the property of being free on a computad. Finally, we show that the constructions of opposites and homs commute.

1 Introduction

In recent years, higher category theory has found a wide range of applications in various fields of mathematics and computer science. Globular higher categories and their computads have been used significantly in rewriting theory [22], homology theory [18], homotopy theory [13] and topological quantum field theory [3]. Furthermore, globular higher groupoids describe identity types in homotopy type theory [2, 21, 25] and they are conjectured to model spaces [14, 15]. Computads play an important role in the homotopy theory of higher categories. They are combinatorial structures that freely generate higher categories, and characterise cofibrant objects in the folk model structure of strict ω -categories [19, 23].

Strict ω -categories are one of the most studied kind of globular higher category. They are usually defined via enrichment, so their composition operations are associative and unital and satisfy the interchange laws. An immediate consequence of their definition is the existence of opposite and hom ω -categories. Opposites are obtained by exchanging the source and target of cells and reversing the order of composition, whereas enrichment gives an ω -category structure to the collection of cells between two objects. While opposites preserve freeness on a computad, the construction of hom ω -categories does not. This is due to the Eckmann-Hilton argument, which imposes a commutativity relation on the arrows of certain hom ω -categories. In this paper, we focus on weak ω -categories and show that the construction of homs does preserve freeness on a computad in this case.

*University of Cambridge, tjb201@cam.ac.uk

†University of Cambridge, ioannis.markakis@cl.cam.ac.uk

Contributions. We give a novel description of the hom functor for weak ω -categories, using the computads-first approach of Dean et al. [11], and we show that it admits a left adjoint, the suspension functor. Moreover, we use the same approach to construct the opposites of ω -categories, and we show that the two constructions commute.

The opposite and the suspension functors preserve freeness on a computad by construction. Our main result is that the same is true for the hom functor. We first identify the generators of the hom ω -category of a computad as its indecomposable cells. Every cell of the hom ω -category can be decomposed into indecomposable cells. Moreover, the generators satisfy no relations, so the hom ω -category of a computad is freely generated by its indecomposable cells.

We further provide an implementation of opposites as a meta-operation in the proof assistant `catt`¹ for ω -categories, based on the type theory of Finster and Mimram [12]. This implementation relies on the equivalence of contexts in this type theory with finite computads [8]. In conjunction with the automatic computation of suspensions [6] and of inverses [7], this meta-operation can be used, for example, to reduce the size of the code that produces cells witnessing the Eckmann-Hilton argument, its inverse and their cancellation witnesses by 95%.

Related work. Both our constructions proceed similarly, taking advantage of the inductive description of computads of Dean et al. [11] to lift adjoints first from globular sets to computads, and then from computads to ω -categories. Those constructions provide further evidence that this computads-first approach is well-suited for developing the theory of ω -categories.

The construction of the suspension functor is inspired by the suspension meta-operation on the type theory `catt`, introduced in the first author's thesis [6]. The former extends the latter via the identification of `catt` contexts with finite computads [8].

Hom ω -categories have been previously constructed by Cottrell and Fujii [10], using Leinster's approach to ω -categories. We give an alternative explicit construction of the hom ω -category functor, that allows us to prove that it preserves the property of being free on a computad. We note that even though both constructions provide an ω -category structure on the same globular set of morphisms, they need not coincide. We conjecture that instead they are merely weakly equivalent.

Overview. We start in Section 2 by recalling the suspension and hom adjunction

$$\Sigma : \text{Glob} \rightleftarrows \text{Glob}^{**} : \Omega$$

between globular sets and bipointed globular sets. We then use this adjunction to describe globular pasting diagrams, the arities of the operations of ω -categories. In Section 3, we introduce computads and weak ω -categories following Dean et

¹Available at <https://github.com/thibautbenjamin/catt>

al. [11]. First, the category of computads and morphisms of free ω -categories is defined by an inductive construction, together with an adjunction with globular sets

$$\text{Free} : \text{Glob} \rightleftarrows \text{Comp} : \text{Cell}.$$

Then, ω -categories are defined as algebras for the monad T induced by this adjunction.

Section 4 is dedicated to the construction of the suspension and hom adjunction. We start by extending the suspension of globular sets to computads in a way that is compatible with the adjunction $\text{Free} \dashv \text{Cell}$. Compatibility with the adjunction gives rise to a natural transformation

$$\Sigma^T : \Sigma T \Rightarrow T^{**} \Sigma$$

where T^{**} is the free bipointed ω -category monad. The mate of this natural transformation is a morphism of monads. Hence, it induces an extension

$$\Omega : \omega \text{Cat}^{**} \rightarrow \omega \text{Cat}$$

of the hom functor to ω -categories by the formal theory of monads [24]. Moreover, by the adjoint lifting theorem, we obtain a left adjoint to this functor.

We then prove our main result that the hom functor preserves freeness on a computad. From an ω -category \mathbb{X} , we construct a computad $C_{\mathbb{X}}$ whose generators are the indecomposable cells of \mathbb{X} together with a morphism of ω -categories $\sigma_{\mathbb{X}} : C_{\mathbb{X}} \rightarrow \mathbb{X}$ from the free ω -category on $C_{\mathbb{X}}$ to \mathbb{X} . We then show that $\sigma_{\mathbb{X}}$ is an isomorphism in the case that \mathbb{X} is the hom ω -category of a computad. It follows that the hom ω -category functor restricts to a functor

$$\Omega : \text{Comp}^{**} \rightarrow \text{Comp}$$

on the level of computads.

In Section 5, we use the same inductive technique to construct the opposites of a weak ω -category. In general, n -categories admit $2^n - 1$ opposites, obtained by reversing the direction of cells of certain dimensions. Similarly, ω -categories have uncountably many opposites, indexed by the group G of subsets of the positive natural numbers. We construct opposite ω -categories by first defining the opposites of globular sets and computads

$$\text{op} : \text{Glob} \rightarrow \text{Glob} \qquad \text{op} : \text{Comp} \rightarrow \text{Comp}$$

in a way that is compatible with the adjunction $\text{Free} \dashv \text{Cell}$. This gives rise to an endomorphism of the free ω -category monad, hence an endofunctor

$$\text{op} : \omega \text{Cat} \rightarrow \omega \text{Cat}$$

of the category of ω -categories. We then show that the formation of opposites gives rise to an action G on the category of ω -categories, so in particular the functors op are involutive. Finally, we show that the formation of opposites and

homs commute by proving a commutativity result for the associated morphisms of monads.

Throughout this paper we illustrate with simple examples how the constructions we introduce interact with the operations of ω -categories. In Section 6, we conclude with a discussion of a more involved example, that of the Eckmann-Hilton cells. We explain how the suspension can be used to raise the dimension of Eckmann-Hilton cells. We remark that certain opposites of Eckmann-Hilton cells are also their inverses, which is proven in our subsequent article [7].

2 Globular pasting diagrams

In this section, we briefly recall the notion of globular pasting diagrams, since they are a basic ingredient for any definition of weak ω -categories. Those are a family of globular sets such that diagrams indexed by them in a strict ω -category can be composed in a unique way. Pasting diagrams are parametrised by rooted, planar trees [5], an inductive description of whose as iterated lists was recently given by Dean et al [11, Section 2]. Our presentation in this section follows ibid. and Leinster [20, Appendix F.2], noting that pasting diagrams are bipointed globular sets generated by the suspension and the wedge sum operations.

To set the notation, we recall that *globular sets* are presheaves on the category \mathbb{G} of globes with objects the natural numbers \mathbb{N} and morphisms freely generated by the source and target inclusions

$$s_n, t_n: [n] \rightarrow [n+1]$$

under the globularity relations:

$$s_{n+1} \circ s_n = t_{n+1} \circ s_n \qquad s_{n+1} \circ t_n = t_{n+1} \circ t_n.$$

In other words, a globular set X consists of a set X_n for every natural number $n \in \mathbb{N}$ together with *source* and *target* functions

$$\text{src}, \text{tgt}: X_{n+1} \rightarrow X_n$$

satisfying the duals relations:

$$\text{src} \circ \text{src} = \text{src} \circ \text{tgt} \qquad \text{tgt} \circ \text{src} = \text{tgt} \circ \text{tgt}.$$

We will call elements of X_n the *n-cells* of X . The *k-source* and *k-target* of an *n-cell* x for $k < n$ are the *k-cells* defined by

$$\text{src}_k x = \text{src}(\cdots(\text{src} x)) \qquad \text{tgt}_k x = \text{tgt}(\cdots(\text{tgt} x))$$

We will denote by \mathbb{D}^n the representable globular set associated to a natural number $n \in \mathbb{N}$, and call it the *n-disk*.

Bipointed globular sets are triples (X, x_-, x_+) consisting of a globular set and two distinguished 0-cells x_-, x_+ of it. They form a category Glob^{**} together

with morphisms of globular sets that preserve the distinguished 0-cells. By the Yoneda lemma, this is the coslice category $\mathbb{D}^0 + \mathbb{D}^0 \backslash \text{Glob}$ or the category of cospans of globular sets from \mathbb{D}^0 to itself.

The category of bipointed globular sets is locally finitely presentable as a coslice of a presheaf topos [1, Proposition 1.57], so in particular it is complete and cocomplete. Limits and connected colimits in Glob^{**} are computed as in Glob , i.e. they are created by the functor $\text{Glob}^{**} \rightarrow \text{Glob}$ that forgets the basepoints. The coproduct of a family of bipointed sets is computed as the wide pushout of the corresponding maps out of $\mathbb{D}^0 + \mathbb{D}^0$.

Being a category of cospans, the category Glob^{**} is monoidal with respect to the composition of cospans, which we will call the *wedge sum*. More explicitly, the wedge sum \vee of a pair of bipointed globular sets (X, x_-, x_+) and (Y, y_-, y_+) is obtained by the following pushout square in Glob ,

$$\begin{array}{ccccc}
 & & X \vee Y & & \\
 & \text{in}_1 \nearrow & \vee & \nwarrow \text{in}_2 & \\
 & X & & Y & \\
 x_- \nearrow & \longleftarrow x_+ & & y_- \longrightarrow & y_+ \\
 \mathbb{D}^0 & & \mathbb{D}^0 & & \mathbb{D}^0
 \end{array}$$

with basepoints the image of x_- and the image of y_+ in the pushout. The unit of the wedge sum is given by the 0-disk \mathbb{D}^0 with both basepoints being its unique 0-cell. More generally, we will denote by $\bigvee_{i=1}^n X_i$ the iterated monoidal product of a finite family of bipointed globular sets X_1, \dots, X_n , and we will denote the inclusion of the j -th component for $1 \leq j \leq n$ by

$$\text{in}_j : X_j \rightarrow \bigvee_{i=1}^n X_i.$$

This is a morphism of globular sets, that is only a morphism of bipointed globular sets for $n = 1$.

The *suspension* of a globular set X is the bipointed globular set ΣX with two 0-cells v_- and v_+ , and with positive dimensional cells given by

$$(\Sigma X)_{n+1} = X_n$$

for every $n \in \mathbb{N}$. The source and target maps of an n -cell of ΣX for $n > 2$ is given by its source and target in X , while the source and target of 1-cells are given by v_- and v_+ respectively. The basepoints of the suspensions are v_- and v_+ . Suspension is left adjoint to the *path space* functor $\Omega : \text{Glob}^{**} \rightarrow \text{Glob}$ sending a bipointed globular set (X, x_-, x_+) to the globular set given by

$$\Omega(X, x_-, x_+)_n = \{x \in X_{n+1} \mid \text{src}_0(x) = x_- \text{ and } \text{tgt}_0(x) = x_+\}$$

The unit of the adjunction is the identity of the functor

$$\Omega \Sigma = \text{id},$$

while the counit $\kappa : \Sigma\Omega \Rightarrow \text{id}$ is the natural transformation with components the bipointed morphisms

$$\kappa : \Sigma\Omega(X, x_-, x_+) \rightarrow (X, x^-, x^+)$$

given by the subset inclusions $\Omega(X, x_-, x_+)_n \subseteq X_{n+1}$.

Finally, we have introduced all the ingredients to define globular pasting diagrams and the family parametrising them. We will call elements of that family Batanin trees following Dean et al [11].

Definition 1. A *Batanin tree* is a list $\text{br}[B_1, \dots, B_n]$, where the B_i are Batanin trees.

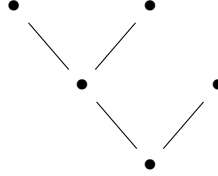
In other words, the set Bat of Batanin trees is the carrier of the initial algebra of the list endofunctor $\text{List} : \text{Set} \rightarrow \text{Set}$ given by

$$\text{List } X = \coprod_{n \in \mathbb{N}} X^n$$

with the obvious action on morphisms. In particular, there exists a tree $\text{br}[]$ corresponding to the empty list, and using this tree, we can define more complicated trees, such as the tree

$$B = \text{br}[\text{br}[\text{br}[], \text{br}[]], \text{br}[]].$$

It is convenient to visualise Batanin trees as planar trees by representing $\text{br}[]$ as a tree with one root and no branches, and $\text{br}[B_1, \dots, B_n]$ as a tree with a new root and n branches, each of which is connected to the root of the tree corresponding to B_i . For example, the tree B above can be visualised as



The *dimension* of a Batanin tree is the height of the corresponding planar tree, or equivalently the maximum of the dimension of the cells in the corresponding globular pasting diagram, defined below. It can be computed recursively by

$$\dim(\text{br}[B_1, \dots, B_n]) = \max(\dim B_1 + 1, \dots, \dim B_n + 1).$$

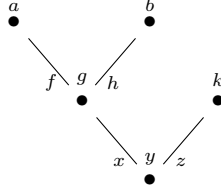
In particular, it follows that $\text{br}[]$ is the unique tree of dimension 0.

Definition 2. The *bipointed globular set of positions* of a Batanin tree B is the bipointed globular set $\text{Pos}(B)$ defined recursively by

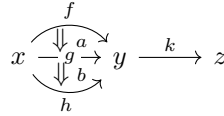
$$\text{Pos}(\text{br}[B_1, \dots, B_n]) = \bigvee_{i=1}^n \Sigma \text{Pos}(B_i).$$

The *globular pasting diagram* of a Batanin tree B is the underlying globular set of $\text{Pos}(B)$, according to the formulae in [20, Appendix F.2].

A way to calculate the globular set of positions of a tree is described in [9], where positions correspond to *sectors* of the tree, i.e. the spaces between two consecutive branches at each node, as well as the space before the first branch and the one after the last one. Under this description, the basepoint are given by the left-most and right-most sector at the root. For the tree B above, we can label the position as follows.



The dimension of a position is given by the distance of the node it is attached in from the root, while its source and target are given by the positions right below it. Therefore, the globular set of positions of B is the following globular set

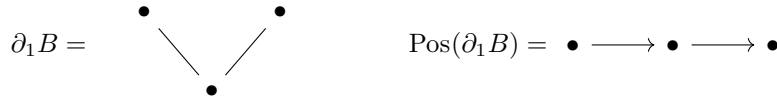


which is bipointed by the positions x and z respectively. Here, the positions f, g, h, a, b are the positions of the left branch of B , while k is the position of its right branch. The dimension of those positions has been raised by the suspension operation. The 0-positions x, y, z are the new cells created by the suspension operation. The two basepoints of $\text{Pos}(B)$ are given by x and z .

Definition 3. The k -boundary of a Batanin tree B is the tree $\partial_k B$ defined recursively by

$$\begin{aligned} \partial_0 B &= \text{br}[] \\ \partial_{k+1} \text{br}[B_1, \dots, B_n] &= \text{br}[\partial_k B_1, \dots, \partial_k B_n] \end{aligned}$$

The k -boundary of a tree B is the tree obtained by removing all nodes of B whose distance from the root is at least k . In terms of pasting diagrams, this amounts to removing all cells of dimension more than k and identifying all parallel k -cells. For example, the 1-boundary of the tree B considered above is the following tree.



The positions of the boundary can be included back into the positions of the original tree in two ways, the *source* and *target inclusions*

$$s_k^B, t_k^B: \text{Pos}(\partial_k B) \rightarrow \text{Pos}(B)$$

defined recursively as follows: the morphisms s_0^B and t_0^B out of $\mathbb{D}^0 \cong \text{Pos}(\text{br}[])$ select the first and second basepoint respectively, while for $B = \text{br}[B_1, \dots, B_n]$ the morphisms s_{k+1}^B and t_{k+1}^B are given by

$$s_{k+1}^B = \bigvee_{i=1}^n \Sigma s_k^{B_i} \quad t_{k+1}^B = \bigvee_{i=1}^n \Sigma t_k^{B_i}.$$

In particular, the source and target inclusions are morphisms of bipointed globular sets when $k > 0$.

As an illustration for Batanin tree and pasting schemes, we consider a few important families of Batanin trees, for which we also illustrate a few of the corresponding globular pasting diagrams for visualisation:

Example 4. We define the k -dimensional disk tree D_k recursively on k by

$$D_0 = \text{br}[] \quad D_{k+1} = \text{br}[D_k].$$

A simple inductive argument shows that $\text{Pos}(D_k) \cong \mathbb{D}^k$.

Example 5. We now define a family of Batanin trees that are the arities of the binary compositions in ω -categories. We define first for $n, m > 0$,

$$B_{n,0,m} = \text{br}[D_n, D_m] \quad \text{Pos}(B_{n,0,m}) \cong \mathbb{D}^n \vee \mathbb{D}^m.$$

A few low-dimensional globular pasting diagrams in this family are illustrated below

$$\begin{array}{ccc} \text{Pos}(B_{1,0,1}) & \text{Pos}(B_{2,0,1}) & \text{Pos}(B_{3,0,3}) \\ \bullet \longrightarrow \bullet \longrightarrow \bullet & \bullet \begin{array}{c} \curvearrowright \\ \Downarrow \\ \curvearrowleft \end{array} \bullet \longrightarrow \bullet & \bullet \begin{array}{c} \Downarrow \Rightarrow \Downarrow \\ \Downarrow \Rightarrow \Downarrow \end{array} \bullet \begin{array}{c} \Downarrow \Rightarrow \Downarrow \\ \Downarrow \Rightarrow \Downarrow \end{array} \bullet \end{array}$$

We then define recursively on k the family

$$B_{n,k,m} = \text{br}[B_{n-1,k-1,m-1}]$$

for every $n, m > 0$ and $0 \leq k < \min(n, m)$. Using that the suspension preserves representables and connected colimits, we conclude that

$$\text{Pos}(B_{n,k,m}) \cong \mathbb{D}^n \amalg_{\mathbb{D}^k} \mathbb{D}^m$$

where the pushout is over the target and source inclusions respectively. We illustrate some of the globular pasting diagrams in this family below.

$$\begin{array}{ccc} \text{Pos}(B_{2,1,2}) & \text{Pos}(B_{3,1,2}) & \text{Pos}(B_{3,2,3}) \\ \bullet \begin{array}{c} \curvearrowright \\ \Downarrow \\ \curvearrowleft \end{array} \bullet & \bullet \begin{array}{c} \curvearrowright \\ \Downarrow \Rightarrow \Downarrow \\ \curvearrowleft \end{array} \bullet & \bullet \begin{array}{c} \Downarrow \Rightarrow \Downarrow \\ \Downarrow \Rightarrow \Downarrow \end{array} \bullet \end{array}$$

A simple induction shows that the dimension of the Batanin tree $B_{n,k,m}$ is given by

$$\dim(B_{n,k,m}) = \max(n, m)$$

3 Computads and ω -categories

Dean et al. [11] recently presented a new definition of ω -categories and their computads, inspired by the type-theoretic definition of Finster and Mimram [12], and they showed that their notion of ω -category coincides with the operadic definition of Leinster [20]. In this approach, first a category of computads Comp is defined together with an adjunction

$$\text{Free}: \text{Glob} \rightleftarrows \text{Comp}: \text{Cell}$$

and then ω -categories are defined as algebras for the monad

$$T: \text{Glob} \rightarrow \text{Glob}$$

induced by the adjunction. We recall that morphisms of computads here are strict ω -functors, and not Batanin's morphisms of computads [4]. In other words, the comparison functor

$$K^T: \text{Comp} \rightarrow \omega \text{Cat}$$

is fully faithful and injective on objects.

We will briefly recall the definition of computads and the $\text{Free} \dashv \text{Cell}$ adjunction. First, categories Comp_n of n -computads are defined recursively for every natural number $n \in \mathbb{N}$, together with forgetful functor

$$u_n: \text{Comp}_n \rightarrow \text{Comp}_{n-1}$$

for $n > 0$. In the same mutual recursion, functors

$$\text{Free}_n: \text{Glob} \rightarrow \text{Comp}_n$$

$$\text{Cell}_n: \text{Comp}_n \rightarrow \text{Set}$$

$$\text{Sph}_n: \text{Comp}_n \rightarrow \text{Set}$$

are defined and natural transformations

$$\text{bdry}_n: \text{Cell}_n \Rightarrow \text{Sph}_n u_n$$

$$\text{pr}_i: \text{Sph}_n \Rightarrow \text{Cell}_n$$

for $i = 1, 2$. Here the functors Cell_n and Sph_n return the set of n -cells, and the set of pairs of parallel n -cells of the ω -category generated by a computad C , while bdry_n returns the source and the target of an n -cell.

An n -computad is a triple C consisting of an $(n-1)$ -computad C_{n-1} , a set of n -dimensional generators V_n^C and an attaching function

$$\phi_n^C: V_n^C \rightarrow \text{Sph}_{n-1}(C_{n-1})$$

assigning to each generator a source and target. A morphism $\sigma: C \rightarrow D$ consists of a morphism $\sigma_{n-1}: C_{n-1} \rightarrow D_{n-1}$ and a function $\sigma_{n,V}: V_n^C \rightarrow \text{Cell}_n D$ compatible with the source and target functions in the sense defined in [11, Section 3.1]. The forgetful functors u_n are the obvious projections. As a base case

for this definition, here we let Comp_{-1} be the terminal category and Sph_{-1} the functor choosing some terminal set.

The set $\text{Cell}_n C$ of n -cells of a computad C is inductively defined together with the set of morphisms with target C and the function $\text{bdry}_{n,C}$. Cells of C are either of the form $\text{var } v$ for a generator $v \in V_n^C$, or when $n > 0$, they are *coherence cells* $\text{coh}(B, A, \tau)$, where B is a tree of dimension at most n , A is an $(n-1)$ -sphere of $\text{Free}_{n-1} \text{Pos}(B)$, satisfying a *fullness* condition that will be explained below, and $\tau: \text{Free}_n \text{Pos}(B) \rightarrow C$ is a morphism. The boundary of a cell is given recursively by the formula

$$\begin{aligned} \text{bdry}_{n,C}(\text{var } v) &= \phi_n^C(v) \\ \text{bdry}_{n,C}(\text{coh}(B, A, \tau)) &= \text{Sph}_{n-1}(\tau_{n-1})(A) \end{aligned}$$

The functor Free_n sends a globular set X to the computad

$$\text{Free}_n X = (\text{Free}_{n-1} X, X_n, \phi_n^X) \quad \phi_n^X(x) = (\text{var}(\text{src } x), \text{var}(\text{tgt } x)),$$

and a morphism $f: X \rightarrow Y$ to the morphism consisting of $\text{Free}_{n-1} f$ and $\text{var} \circ f_n$.

The functor Sph_n sends an n -computad C to the set

$$\text{Sph}_n C = \{(a, b) \in \text{Cell}_n C \times \text{Cell}_n C \mid \text{bdry}_n a = \text{bdry}_n b\}$$

and acts on morphisms in the obvious way. The projection natural transformations are the obvious ones. We will denote by

$$\text{src, tgt}: \text{Cell}_n \Rightarrow \text{Cell}_{n-1} u_n$$

the composite of bdry_n with the projections.

The fullness condition mentioned above for $A = (a, b) \in \text{Sph}_n \text{Free}_n \text{Pos}(B)$ is a condition on the generators used to define a and b . It is equivalent to the statement that

$$a = \text{Cell}_n \text{Free}_n(s_n^B)(a') \quad b = \text{Cell}_n \text{Free}_n(t_n^B)(b')$$

for cells a', b' of $\text{Free}_n \text{Pos}(\partial_n B)$ using all generators of $\partial_n B$. That means that the *support* of a', b' contains all positions of $\partial_n B$, where the support of an n -cell c over a computad C is the set of generators defined by

$$\begin{aligned} \text{supp}(\text{var } v) &= \begin{cases} \{v\}, & \text{when } n = 0 \\ \{v\} \cup \text{supp}(\text{pr}_1 \phi_n^C(v)) \cup \text{supp}(\text{pr}_2 \phi_n^C(v)), & \text{when } n > 0 \end{cases} \\ \text{supp}(\text{coh}(B, A, \tau)) &= \bigcup_{k \leq n} \bigcup_{v \in \text{Pos}_k(B)} \text{supp}(\tau_{k,V}(v)) \end{aligned}$$

This completes the inductive definition. The category Comp of computads is the limit of the categories Comp_n for all $n \in \mathbb{N}$, i.e. computads $C = (C_n)_{n \in \mathbb{N}}$ are sequences of n -computads C_n such that $u_{n+1} C_{n+1} = C_n$, and morphisms of such are sequences of morphisms. The free functor

$$\text{Free}: \text{Glob} \rightarrow \text{Comp}$$

is the functor with components Free_n for all $n \in \mathbb{N}$, while the cell functor

$$\text{Cell}: \text{Comp} \rightarrow \text{Glob}$$

sends a computad C to the globular set consisting of $\text{Cell}_n C_n$ for all $n \in \mathbb{N}$, and the source and target functions defined above. The unit and the counit of the adjunction

$$\eta: \text{id} \Rightarrow \text{Cell Free}$$

$$\varepsilon: \text{Free Cell} \Rightarrow \text{id}$$

are described as follows: The unit η sends a cell x of a globular set X to the generator $\text{var } x$, the counit ε consists of the morphisms $\varepsilon_C: \text{Free Cell } C \rightarrow C$ given for all $n \in \mathbb{N}$ by the identities of the set

$$V_n^{\text{Free Cell } C} = \text{Cell}_n C.$$

By definition, ω -categories are algebras for the monad (T, μ, η) on Glob induced by the adjunction $\text{Free} \dashv \text{Cell}$. In particular, there exists a free/underlying adjunction

$$F^T: \text{Glob} \rightleftarrows \omega \text{Cat}: U^T$$

between globular sets and ω -categories, and there exists a comparison functor

$$K^T: \text{Comp} \rightarrow \omega \text{Cat}$$

sending a computad C to the ω -category $(\text{Cell } C, \text{Cell } \varepsilon_C)$. Moreover, K^T is a morphism of adjunctions meaning that

$$F^T = K^T \text{Free} \quad \text{Cell} = U^T K^T,$$

and it is fully faithful [11, Proposition 4.6].

3.1 Binary compositions in ω -categories

The aim of this section is to illustrate this definition by describing the binary compositions of ω -categories. To achieve this, we construct a cell $\text{comp}_{n,k,m}$ of dimension $\max(n, m)$ in the free ω -category on the globular pasting diagram $\text{Pos}(B_{n,k,m})$, introduced in Example 5. By the pushout formulae in Example 5, morphisms of ω -categories $f: T \text{Pos}(B_{n,k,m}) \rightarrow \mathbb{X}$ correspond precisely to pairs of an n -cell c and an m -cell c' in \mathbb{X} such that $\text{tgt}_k(c) = \text{src}_k(c')$. The k -composite of c and c' is then defined to be the cell

$$c *_k c' = f(\text{comp}_{n,k,m}).$$

The rest of the operations of ω -categories can be obtained in a similar way, by defining specific cells in well-chosen computads. This is in particular the case for associators, unitors and interchangers. We now define the cells $\text{comp}_{n,k,m}$ starting from simpler cases and building up towards the general case.

We first define the cell $\text{comp}_{1,0,1}$. By the pushout formula, we visualise the globular pasting diagram $\text{Pos}(B_{1,0,1})$ as follows:

$$x \xrightarrow{f} y \xrightarrow{g} z ,$$

we then define the 1-cell of $\text{Free Pos}(B_{1,0,1})$

$$\text{comp}_{1,0,1} := \text{coh}(B_{1,0,1}, (\text{var } x, \text{var } z), \text{id}),$$

with source $\text{var } x$ and target $\text{var } z$. Similarly, we use that $\partial_k B_{k+1,k,k+1} = D_k$ to define the cell

$$\text{comp}_{k+1,k,k+1} := \text{coh}(B_{k+1,k,k+1}, (\text{var } x_{k+1}, \text{var } z_{k+1}), \text{id}),$$

where x_{k+1} and z_{k+1} are respectively the images of the top-dimensional cell of the disk $\text{Pos}(D_k)$ under the source and target inclusions $\text{Pos}(D_k) \rightarrow \text{Pos}(B_{k+1,k,k+1})$ defined in Section 2. The source and target of $\text{comp}_{k+1,k,k+1}$ imply that the source of $c *_k c'$ is $\text{src}(c)$ and its target is $\text{tgt}(c')$, as expected of composition operations.

We then proceed to define the n -cell $\text{comp}_{n,k,n}$ recursively on $n - k$. The base case $n = k + 1$ has been defined above. For $n > k + 1$, we have that $\partial_{n-1} B_{n,k,n} = B_{n-1,k,n-1}$. We then define the cell $\text{comp}_{n,k,n}$ to be

$$\text{coh}(B_{n,k,n}, (T(s_{n-1}^{B_{n,k,n}})(\text{comp}_{n-1,k,n-1}), T(t_{n-1}^{B_{n,k,n}})(\text{comp}_{n-1,k,n-1})), \text{id}).$$

It follows from this definition that the source and target of the k -composite $c *_k c'$ are respectively the k -composite of the sources and the k -composite of the targets.

Finally, we define the cell $\text{comp}_{n,k,m}$ for arbitrary $n \neq m$ and $k < \min(n, m)$. When $n > m$, we have that $\partial_{n-1} B_{n,k,m} = B_{n-1,k,m}$, and we define recursively the n -cell $\text{comp}_{n,k,m}$ to be

$$\text{coh}(B_{n,k,m}, (T(s_{n-1}^{B_{n,k,m}})(\text{comp}_{n-1,k,m}), T(t_{n-1}^{B_{n,k,m}})(\text{comp}_{n-1,k,m})), \text{id}).$$

When $n < m$, we construct the m -cell $\text{comp}_{n,k,m}$ in a similar way. The binary k -composite $c *_k c'$ is the whiskering of an n -cell with an m -cell. Its source and target are the k -composites of the source and target of the higher dimensional cell with the lower dimensional one. In the following sections, we will see that leveraging the suspension and opposite operations, the construction of the cells $\text{comp}_{n,k,m}$ can be significantly simplified.

4 The suspension and hom functors

Strict ω -categories are precisely categories enriched over strict ω -categories, so for every strict ω -category X and every pair of 0-cells $x_-, x_+ \in X_0$, the globular set $\Omega(X, x_-, x_+)$ of cells from x_- to x_+ admits an ω -category structure in a functorial way [20]. The same result was recently proven for arbitrary ω -categories [10]

using the operadic definition of Leinster. We provide an elementary construction of hom ω -categories, and show that it preserves the property of being free on a computad. This diverges from strict ω -categories, where the Eckmann-Hilton argument is an obstruction to freeness of hom ω -categories.

Recall that the *hom* functor $\Omega : \text{Glob}^{**} \rightarrow \text{Glob}$, taking a bipointed globular set to the globular set of cells from the first base-point to the second one admits a left adjoint, the *suspension* functor, $\Sigma : \text{Glob} \rightarrow \text{Glob}^{**}$ with unit the identity and counit $\kappa : \Sigma \Omega \Rightarrow \text{id}$ given by subset inclusions. Our goal in this section will be to lift this adjunction to computads and to ω -categories.

4.1 The suspension of a computad

Let Comp^{**} be the category of computads with two chosen 0-cells and morphisms preserving those 0-cells. By the $\text{Free} \dashv \text{Cell}$ adjunction and the Yoneda lemma, this is precisely the slice of Comp under $\text{Free}(\mathbb{D}^0 + \mathbb{D}^0)$. Moreover, the adjunction descends to the slices to give an adjunction

$$\text{Free}^{**} : \text{Glob}^{**} \rightleftarrows \text{Comp}^{**} : \text{Cell}^{**}$$

where $\text{Free}^{**}(X, x_-, x_+)$ is the computad $\text{Free} X$ with the 0-cells $\text{var } x_-$ and $\text{var } x_+$, and $\text{Cell}^{**}(C, c_-, c_+)$ is the globular set $\text{Cell} C$ with the basepoints c_- and c_+ respectively. We now define a suspension functor

$$\Sigma : \text{Comp} \rightarrow \text{Comp}^{**}$$

which generalises the suspension of globular sets.

We define first the *suspension* of a Batanin tree B to be the Batanin tree

$$\Sigma B = \text{br}[B]$$

since by definition

$$\text{Pos}(\Sigma B) = \Sigma \text{Pos}(B).$$

We will then proceed inductively on $n \in \mathbb{N}$ to define a functor and two natural transformations

$$\begin{aligned} \Sigma &: \text{Comp}_n \rightarrow \text{Comp}_{n+1} \\ \Sigma^{\text{Cell}} &: \text{Cell}_n \Rightarrow \text{Cell}_{n+1} \Sigma \\ \Sigma^{\text{Sph}} &: \text{Sph}_n \Rightarrow \text{Sph}_{n+1} \Sigma \end{aligned}$$

satisfying the following properties:

- (S1) the suspension commutes with the forgetful functors, and the inclusion of globular sets into computads:

$$\begin{array}{ccc} \text{Comp}_{n+1} & \xrightarrow{\Sigma} & \text{Comp}_{n+2} \\ \downarrow u_{n+1} & & \downarrow u_{n+2} \\ \text{Comp}_n & \xrightarrow{\Sigma} & \text{Comp}_{n+1} \end{array} \qquad \begin{array}{ccc} \text{Comp}_n & \xrightarrow{\Sigma} & \text{Comp}_{n+1} \\ \text{Free}_n \uparrow & & \uparrow \text{Free}_{n+1} \\ \text{Glob} & \xrightarrow{\Sigma} & \text{Glob} \end{array}$$

- (s2) the natural transformations are compatible with the boundary natural transformations:

$$\begin{array}{ccc}
\text{Cell}_{n+1} & \xrightarrow{\Sigma^{\text{Cell}}} & \text{Cell}_{n+2} \Sigma \\
\text{bdry}_{n+1} \Downarrow & & \Downarrow \text{bdry}_{n+1} \Sigma \\
\text{Sph}_n u_{n+1} & \xrightarrow{\Sigma^{\text{Sph}}_{u_{n+1}}} & \text{Sph}_{n+1} \Sigma u_{n+1} \xlongequal{\quad} \text{Sph}_{n+1} u_{n+1} \Sigma
\end{array}$$

- (s3) the natural transformations are compatible with the projection natural transformations for $i = 1, 2$:

$$\begin{array}{ccc}
\text{Sph}_n & \xrightarrow{\Sigma^{\text{Sph}}} & \text{Sph}_{n+1} \Sigma \\
\Downarrow \text{pr}_i & & \Downarrow \text{pr}_i \Sigma \\
\text{Cell}_n & \xrightarrow{\Sigma^{\text{Cell}}} & \text{Cell}_{n+1} \Sigma
\end{array}$$

- (s4) the natural transformation Σ^{Cell} preserves generators, in that for every globular set X and $x \in X_n$, we have that

$$\Sigma^{\text{Cell}}(\text{var } x) = \text{var } x$$

- (s5) the natural transformation Σ^{Sph} preserves fullness, in that for every full n -sphere A of $\text{Free Pos}(B)$, we have that $\Sigma^{\text{Sph}} A$ is a full $(n+1)$ -sphere of $\text{Free Pos}(\Sigma B)$.

To start the induction, we recall that Comp_{-1} is the terminal category and that Comp_0 is the category Set of sets. The suspension functor is defined as the functor picking the 2-element set $\{v_-, v_+\}$. We recall also that the unique (-1) -computad has a unique (-1) -sphere. We define Σ^{Sph} to be the natural transformation picking the 0-sphere (v_-, v_+) . This concludes the base case. We will now assume that we have defined the the data satisfying the properties we have cited, up to dimension $n-1$, for a fixed $n \in \mathbb{N}$.

Computads. We will first define the functor Σ on all objects: Given an n -computad $C = (C_{n-1}, V_n^C, \phi_n^C)$, its suspension is the $(n+1)$ -computad consisting of ΣC_{n-1} , the same set of generators, and the attaching function $\phi_{n+1}^{\Sigma C}$ given by the composite

$$\phi_{n+1}^{\Sigma C} : V_n^C \xrightarrow{\phi_n^C} \text{Sph}_{n-1} C_{n-1} \xrightarrow{\Sigma^{\text{Sph}}} \text{Sph}_n \Sigma C_{n-1}$$

By definition, the suspension functor on objects commutes with the forgetful functors. Using that the suspension functor on $(n-1)$ -computads commutes also with the inclusion of globular sets, and that Σ^{Cell} and hence Σ^{Sph} preserve generators, we see that the suspension functor on n -computads also commutes with the inclusions.

Cells and morphisms. We then define the suspension of a morphism of n -computads together with the natural transformation Σ^{Cell} mutually inductively, while showing that Property (s2) holds. Given a computad C , we define mutually recursively Σ_C^{Cell} and the morphism and $\Sigma(\sigma)$ for every σ with target C . For a generator $v \in V_n^C$, we let

$$\Sigma_C^{\text{Cell}}(\text{var } v) = \text{var } v,$$

and we compute that

$$(\text{bdry}_{n+1, \Sigma C} \Sigma_C^{\text{Cell}})(\text{var } v) = \phi_{n+1}^{\Sigma C}(v) = (\Sigma_{C_{n-1}}^{\text{Sph}} \text{bdry}_{n, C})(\text{var } v).$$

For a coherence n -cell $c = \text{coh}(B, A, \tau)$ of C , we let

$$\Sigma_C^{\text{Cell}}(\text{coh}(B, A, \tau)) = \text{coh}(\Sigma B, \Sigma^{\text{Sph}} A, \Sigma \tau)$$

using that the suspension commutes with Free_n and $\text{Pos}(-)$, and that it preserves fullness. Then by naturality of Σ^{Sph} , we compute that

$$(\text{bdry}_{n+1, \Sigma C} \Sigma_C^{\text{Cell}})(\text{coh}(B, A, \tau)) = (\Sigma_{C_{n-1}}^{\text{Sph}} \text{bdry}_{n, C})(\text{coh}(B, A, \tau)).$$

For a morphism $\sigma : D \rightarrow C$, we let $\Sigma \sigma : \Sigma D \rightarrow \Sigma C$ consist of $\Sigma \sigma_{n-1}$ and the function

$$(\Sigma \sigma)_V : V_{n+1}^{\Sigma D} = V_n^D \xrightarrow{\sigma_V} \text{Cell}_n(C) \xrightarrow{\Sigma_C^{\text{Cell}}} \text{Cell}_{n+1} \Sigma C$$

This is a well-defined morphism by the observation on the boundary of Σ^{Cell} . Functoriality of the suspension, and naturality of Σ^{Cell} can be shown mutually inductively.

Spheres. Finally, the natural transformation Σ^{Sph} is defined for a computad C and an n -sphere $A = (a, b)$ of it again by

$$\Sigma^{\text{Sph}}(a, b) = (\Sigma^{\text{Cell}} a, \Sigma^{\text{Cell}} b).$$

We observe that those $(n+1)$ -cells are parallel again by Property (s2).

Fullness. To finish the induction, it remains to show that for every Batanin tree B and full n -sphere $A = (a, b)$ of $\text{Free}_n \text{Pos}(B)$, the sphere $\Sigma^{\text{Sph}} A$ is full in $\text{Free}_{n+1} \text{Pos}(\Sigma B)$. To show that, we first let

$$a = \text{Cell}_n \text{Free}_n(s_n^B)(a_0) \quad b = \text{Cell}_n \text{Free}_n(t_n^B)(b_0),$$

where the support of a_0 and b_0 contains all positions of $\partial_n B$. Then $\Sigma^{\text{Sph}} A$ consists of the cells

$$\begin{aligned} a' &= \Sigma^{\text{Cell}}(\text{Cell}_n \text{Free}_n(s_n^B)(a_0)) \\ &= \text{Cell}_n \Sigma \text{Free}_n(s_n^B)(\Sigma^{\text{Cell}}(a_0)) \\ &= \text{Cell}_n \text{Free}_n(\Sigma s_n^B)(\Sigma^{\text{Cell}}(a_0)) \\ &= \text{Cell}_n \text{Free}_n(s_{n+1}^{\Sigma B})(\Sigma^{\text{Cell}}(a_0)) \\ b' &= \text{Cell}_n \text{Free}_n(t_{n+1}^{\Sigma B})(\Sigma^{\text{Cell}}(b_0)), \end{aligned}$$

so it remains to show that when the support of a contains all positions of a tree B , then the support of $\Sigma^{\text{Cell}} a$ contains all positions of ΣB . More generally, it suffices to prove that for every computad C and every cell $c \in \text{Cell}_n C$,

$$\text{supp}(\Sigma^{\text{Cell}}(c)) = \text{supp}(c) \cup \{v_-, v_+\}.$$

This statement can be easily shown by structural induction on cells.

Infinite-dimensional computads. This concludes the induction on $n \in \mathbb{N}$. Compatibility of the suspension functors with the forgetful functors allows us to define a functor

$$\Sigma : \text{Comp} \rightarrow \text{Comp}^{**}$$

sending a computad $C = (C_n)_{n \in \mathbb{N}}$ to the computad with components

$$(\Sigma C)_0 = \{v_-, v_+\} \quad (\Sigma C)_{n+1} = \Sigma C_n,$$

and with basepoints $\text{var } v_-$ and $\text{var } v_+$. By construction, this functor commutes with the suspension operation on globular sets, in that the following square commutes:

$$\begin{array}{ccc} \text{Comp} & \xrightarrow{\Sigma} & \text{Comp}^{**} \\ \text{Free} \uparrow & & \uparrow \text{Free}^{**} \\ \text{Glob} & \xrightarrow{\Sigma} & \text{Glob}^{**} \end{array}$$

Moreover, Property (s2) shows that the natural transformations Σ^{Cell} can be combined to a natural transformation

$$\Sigma^{\text{Cell}} : \Sigma \text{Cell} \Rightarrow \text{Cell}^{**} \Sigma.$$

Example 6. The suspension of the tree $B_{n,k,m}$ of Example 5 is the tree $\Sigma B_{n,k,m} = B_{n+1,k+1,m+1}$. Moreover, applying the natural transformation Σ^{Cell} to the cell $\text{comp}_{n,k,m}$ defined in Section 3.1 yields the cell

$$\Sigma^{\text{Cell}}(\text{comp}_{n,k,m}) = \text{comp}_{n+1,k+1,m+1}.$$

It follows that any cell of the form $\text{comp}_{n,k,m}$ can be obtained by iteratively suspending one of the form $\text{comp}_{r,0,s}$.

4.2 Hom ω -categories

We can define bipointed ω -categories similarly to bipointed globular sets and bipointed computads. Alternatively, the adjunction

$$\text{Free}^{**} : \text{Glob}^{**} \rightleftarrows \text{Comp}^{**} : \text{Cell}^{**}$$

gives rise to a monad T^{**} on bipointed globular sets, sending a bipointed globular set (X, x_-, x_+) to the globular set TX with basepoints $\text{var } x_-$ and $\text{var } x_+$. It

can be shown that its category of algebras is the category ωCat^{**} of bipointed ω -categories.

Whiskering the natural transformation Σ^{Cell} on the right with Free and using compatibility of the suspension with the functor Free , we get a natural transformation

$$\Sigma^T : \Sigma T \Rightarrow T^{**} \Sigma.$$

Equivalently, by the mate correspondence [17, Proposition 2.1], we get a natural transformation

$$\Omega^T = (\Omega T^{**} \kappa) \circ (\Omega \Sigma^T \Omega) : T \Omega \Rightarrow \Omega T^{**}.$$

We will show that this (Ω, Ω^T) is a morphism of monads, meaning that the following diagrams commute:

$$\begin{array}{ccc} T \Omega & \xrightarrow{\Omega^T} & \Omega T^{**} \\ \eta \Omega \swarrow & & \searrow \Omega \eta \\ & \Omega & \end{array} \quad \begin{array}{ccccc} TT \Omega & \xrightarrow{T \Omega^T} & T \Omega T^{**} & \xrightarrow{\Omega^T T^{**}} & \Omega T^{**} T^{**} \\ \mu \Omega \Downarrow & & & & \Downarrow \Omega \mu \\ T \Omega & \xrightarrow{\Omega^T} & \Omega T^{**} & & \end{array}$$

By the mate correspondence, commutativity of those diagrams is equivalent to the commutativity of the following ones:

$$\begin{array}{ccc} \Sigma T & \xrightarrow{\Sigma^T} & T^{**} \Sigma \\ \Sigma \eta \swarrow & & \searrow \eta \Sigma \\ & \Sigma & \end{array} \quad \begin{array}{ccccc} \Sigma TT & \xrightarrow{\Sigma^T T} & T^{**} \Sigma T & \xrightarrow{T^{**} \Sigma^T} & T^{**} T^{**} \Sigma \\ \Sigma \mu \Downarrow & & & & \Downarrow \mu \Sigma \\ \Sigma T & \xrightarrow{\Sigma^T} & T^{**} \Sigma & & \end{array}$$

The left one commutes, since Σ^{Cell} preserves generators. The right one is obtained from the following diagram by whiskering on the right with Free :

$$\begin{array}{ccc} \Sigma T \text{Cell} \xrightarrow{\Sigma^{\text{Cell}} \text{Free Cell}} T^{**} \Sigma \text{Cell} \xrightarrow{T^{**} \Sigma^{\text{Cell}}} T^{**} \text{Cell}^{**} \Sigma \\ \Sigma \text{Cell} \varepsilon \Downarrow \qquad \qquad \qquad \text{Cell}^{**} \Sigma \varepsilon \searrow \qquad \qquad \qquad \Downarrow \text{Cell}^{**} \varepsilon \Sigma \\ \Sigma \text{Cell} \xrightarrow{\Sigma^{\text{Cell}}} \text{Cell}^{**} \Sigma \end{array} \quad (1)$$

This diagram commutes by naturality of Σ^{Cell} and commutativity of the following diagram

$$\begin{array}{ccc} \Sigma \text{Free Cell} & \xrightarrow{\Sigma \varepsilon} & \Sigma \\ \parallel & & \varepsilon \Sigma \Uparrow \\ \text{Free}^{**} \Sigma \text{Cell} & \xrightarrow{\text{Free}^{**} \Sigma^{\text{Cell}}} & \text{Free}^{**} \text{Cell}^{**} \Sigma \end{array}$$

This can be checked easily by showing that both sides agree on generators.

Definition 7. Consider a bipointed ω -category $(X, \alpha : TX \rightarrow X, x_-, x_+)$. Its *hom ω -category* is the ω -category that consists of the globular set $\Omega(X, x, x_+)$ and the structure morphism

$$T \Omega(X, x, x_+) \xrightarrow{\Omega^T} \Omega T^{**} X \xrightarrow{\Omega \alpha} \Omega(X, x, x_+)$$

As shown by Street [24], and explained by Leinster [20, Theorem 6.1.1], this definition extends to a functor

$$\Omega : \omega \text{ Cat}^{**} \rightarrow \omega \text{ Cat},$$

since (Ω, Ω^T) is a morphism of monads. By construction, this functor makes the following solid square commute:

$$\begin{array}{ccc} \omega \text{ Cat}^{**} & \xrightarrow{\Omega} & \omega \text{ Cat} \\ U^{T**} \downarrow & & \downarrow U^T \\ \text{Glob}^{**} & \xrightarrow{\Omega} & \text{Glob} \end{array}$$

$\xleftarrow{\Sigma}$ (top arrow) and $\xleftarrow{\Sigma}$ (bottom arrow)

Example 8. In light of the ω -category structure on the hom globular set, the last equation of Example 6 can be understood in particular as saying that the vertical composite of two 2-cells x, y in an ω -category \mathbb{X} can be described as the composite of the 1-cells x, y in the hom ω -category $\Omega(\mathbb{X}, \text{src } x, \text{tgt } x)$. Similar reasoning also shows that the associators and unitors for the composition of 1-cells in the hom ω -category $\Omega(\mathbb{X}, \text{src } x, \text{tgt } x)$ yields through the natural transformation Σ^{Cell} associators and unitors for the vertical composition in \mathbb{X} .

Suspension. The category of bipointed ω -categories is cocomplete being the category of algebras of a finitary monad on a locally presentable category, and the vertical functors in this commuting square are monadic. Therefore, by the adjoint lifting theorem [16], the hom ω -category functor also admits a left adjoint Σ that commutes with the free functors F^{T**} and F^T . We will show that restricting the left adjoint to the subcategory of computads gives the suspension functor defined before, in the sense that the following diagram commutes up to isomorphism:

$$\begin{array}{ccc} \omega \text{ Cat}^{**} & \xrightarrow{\Sigma} & \omega \text{ Cat} \\ K^{T**} \uparrow & & \uparrow K^T \\ \text{Comp}^{**} & \xrightarrow{\Sigma} & \text{Comp} \end{array}$$

Since the suspension of an ω -category is defined as a left adjoint, commutativity of this square amounts to showing for every computad C and every bipointed ω -category (Y, y_-, y_+) that there exists a natural bijection

$$\psi : \omega \text{ Cat}^{**}(K^{T**}(\Sigma C), Y) \xrightarrow{\sim} \omega \text{ Cat}(K^T C, \Omega Y)$$

We claim that such a bijection is given by

$$\psi(f) = \Omega(f \circ \Sigma_C^{\text{Cell}}) : K^T C \rightarrow \Omega Y$$

First of all, we need to check that $\psi(f)$ is a morphism of ω -categories for every morphism of bipointed ω -categories f . This amounts to commutativity of the

exterior of the following diagram

$$\begin{array}{ccccccc}
T \text{Cell } C & \xlongequal{\quad} & T \Omega \Sigma \text{Cell } C & \xrightarrow{T \Omega \Sigma^{\text{Cell}}} & T \Omega \text{Cell}^{**} \Sigma C & \xrightarrow{T \Omega f} & T \Omega Y \\
\downarrow \text{Cell}(\varepsilon_C) & & & & \Omega^T \downarrow & & \downarrow \Omega^T \\
& & & & \Omega T^{**} \text{Cell}^{**} \Sigma C & \xrightarrow{\Omega T^{**} f} & \Omega T^{**} Y \\
& & & & \Omega \text{Cell}^{**}(\varepsilon_{\Sigma C}) \downarrow & & \downarrow \\
\text{Cell } C & \xlongequal{\quad} & \Omega \Sigma \text{Cell } C & \xrightarrow{\Omega \Sigma^{\text{Cell}}} & \Omega \text{Cell}^{**} \Sigma C & \xrightarrow{\Omega f} & \Omega Y
\end{array}$$

where the unnamed morphism $T^{**}Y \rightarrow Y$ is the T^{**} -algebra structure of Y . The left square in this diagram commutes, being the transpose of diagram (1). The top square commutes by naturality, and the bottom square commutes by f being an algebra morphism. Therefore, $\psi(f)$ is well-defined.

To show that ψ is bijective, we use that morphisms out of a computad are determined by their values on generators [11, Corollary 6.5]. Assume that $\psi(f) = \psi(g)$ for a pair of morphisms of bipointed ω -categories. Then f and g agree on the 0-dimensional generators being bipointed, and they agree on positive-dimensional ones by $\Omega \Sigma^{\text{Cell}}$ being a bijection on the generators. Therefore, $f = g$. For surjectivity, given a morphism $h : K^T C \rightarrow \Omega Y$, we define inductively a sequence of morphisms $f_n : K_n^T(\Sigma C) \rightarrow Y$ by letting f_0 being the unique morphism determined by $f(v_{\pm}) = y_{\pm}$ and f_{n+1} being the morphism induced by f_n and the function

$$V_{n+1}^{\Sigma C} = V_n^C \xrightarrow{\text{var}} \text{Cell}_n C \xrightarrow{h} (\Omega Y)_n \hookrightarrow Y_{n+1}$$

Finally, we let $f : K^T(\Sigma C) \rightarrow Y$ be the morphisms out of the colimits induced by the f_n . Then by construction $\psi(f)$ and h agree on generators, so they must be equal.

4.3 The hom of a computad.

In this section, we show that the hom ω -category functor preserves the property of being free on a computad. This is contrary to the case of strict ω -categories: consider for instance the strict ω -category freely generated on the computad C_{eh} with only one object x , and two generators $a, b : \text{id } x \rightarrow \text{id } x$ of dimension 2. By the Eckmann-Hilton argument, the composition of 1-cells in $\Omega(C_{\text{eh}}, x, x)$ is commutative, which prevents it from being free on a computad.

We will show that $\Omega K^{T^{**}} C$ is free on a computad when C is a bipointed computad for weak ω -categories. To do so, we construct a computad ΩC on which it is free. The generators of ΩC are precisely the *indecomposable cells* of the ω -category $\Omega K^T C$:

Definition 9. An n -cell of an ω -category $\mathbb{X} = (X, \alpha : TX \rightarrow X)$ is *indecomposable* when it is not of the form $\alpha(\text{coh}(B, A, \text{Free } \sigma))$ for any Batanin tree B , full $(n-1)$ -type A and morphism $\sigma : \text{Pos}(B) \rightarrow X$. We will denote the set of indecomposable n -cells by X_n^{ind} .

Using the indecomposable cells of an ω -category \mathbb{X} , we can build a computad $C_{\mathbb{X}}$ together with a morphism $\sigma_{\mathbb{X}} : K^T C_{\mathbb{X}} \rightarrow \mathbb{X}$. We start inductively by letting $C_{\mathbb{X},-1}$ the unique -1 -computad and $\sigma_{\mathbb{X},-1} : K_{-1}^T C_{\mathbb{X},-1} \rightarrow \mathbb{X}$ the unique morphism from the initial ω -category. We then define the n -computad $C_{\mathbb{X},n}$ to be the triple $(C_{\mathbb{X},n-1}, V_n^{\mathbb{X}}, \phi_n^{\mathbb{X}})$ defined via the following pullback square

$$\begin{array}{ccc}
V_n^{\mathbb{X}} & \xrightarrow{\sigma_{\mathbb{X},n,V}^{\text{ind}}} & X_n^{\text{ind}} \\
\downarrow \phi_n^{\mathbb{X}} & \lrcorner & \downarrow \\
& & X_n \\
& & \downarrow (\text{src,tgt}) \\
\text{Sph}_{n-1}(C_{\mathbb{X},n-1}) & \xrightarrow{(\sigma_{\mathbb{X},n-1}, \sigma_{\mathbb{X},n-1})} & \text{Par}_{n-1}(X)
\end{array} \tag{2}$$

where $\text{Par}_{n-1} X$ is the set of pairs of parallel $(n-1)$ -cells of X . The morphism $\sigma_{\mathbb{X},n} : K_n^T C_{\mathbb{X},n} \rightarrow \mathbb{X}$ is the one determined by $\sigma_{\mathbb{X},n-1}$ and the function $V_n^{\mathbb{X}} \rightarrow X_n$ in the diagram above. Finally, we define $C_{\mathbb{X}}$ to consist of the computads $C_{\mathbb{X},n}$, and we let $\sigma_{\mathbb{X}}$ to be the morphism out of the colimit induced by the morphisms $\sigma_{\mathbb{X},n}$.

Proposition 10. *For every bipointed computad C , the ω -category $\Omega K^{T^{**}} C$ is free on a computad.*

Proof. Let $\mathbb{X} = \Omega K^{T^{**}} C$. We will show that the morphism $\sigma_{\mathbb{X}}$ is an isomorphism, or equivalently that it is bijective on all cells. We proceed by strong induction on the dimension of cells. Let therefore $n \geq 0$ and suppose that the result holds for all $k < n$. Since the inclusion $K_{n-1}^T C_{n-1} \rightarrow K^T C$ is the identity on cells of dimension at most $(n-1)$, the bottom morphism of the square (2) must be an isomorphism, and hence the top one should be as well. This implies that $\sigma_{\mathbb{X}}$ is injective on generator n -cells.

This shows that $\sigma_{\mathbb{X}}$ is injective on 0-cells. If $n > 0$, then we can see that $\sigma_{\mathbb{X}}$ sends a coherence n -cell $c = \text{coh}(B, A, \tau)$ to the decomposable n -cell:

$$\begin{aligned}
\sigma_{\mathbb{X}}(c) &= \sigma_{\mathbb{X}}(\text{coh}(B, A, \varepsilon \circ \text{Free } \tau^\dagger)) \\
&= (\sigma_{\mathbb{X}} \circ \text{Cell}(\varepsilon))(\text{coh}(B, A, \text{Free } \tau^\dagger)) \\
&= (\alpha_{\mathbb{X}} \circ \text{Cell Free } \sigma_{\mathbb{X}})(\text{coh}(B, A, \text{Free } \tau^\dagger)) \\
&= \alpha_{\mathbb{X}}(\text{coh}(B, A, \text{Free}(\sigma_{\mathbb{X}} \tau^\dagger)))
\end{aligned}$$

where $\tau^\dagger : \text{Pos}(B) \rightarrow \text{Cell Free } C$ the transpose of τ under the $\text{Free} \dashv \text{Cell}$ adjunction. As $\sigma_{\mathbb{X}}$ sends generators to indecomposable cells, it can not send a generator and a coherence to the same cell.

Finally, to show that $\sigma_{\mathbb{X}}$ is injective on n -cells, it remains to show that it is injective on coherence cells. For that, let $c = \text{coh}(B, A, \tau)$ and $c' = \text{coh}(B', A', \tau')$ and suppose that

$$\sigma_{\mathbb{X}}(c) = \sigma_{\mathbb{X}}(c').$$

From the definition of $\alpha_{\mathbb{X}}$, we can compute further that

$$\begin{aligned}
\sigma_{\mathbb{X}}(c) &= (\Omega \text{Cell}^{**}(\varepsilon_C)(\Omega T^{**} \kappa_{\text{Cell}^{**} C})(\Omega \Sigma_{\Omega \text{Cell}^{**} C}^T))(\text{coh}(B, A, \text{Free}(\sigma_{\mathbb{X}} \tau^\dagger))) \\
&= (\Omega \text{Cell}^{**}(\varepsilon_C)(\Omega T^{**} \kappa_{\text{Cell}^{**} C})(\text{coh}(\Sigma B, \Sigma^{\text{Sph}} A, \Sigma \text{Free}(\sigma_{\mathbb{X}} \tau^\dagger)))) \\
&= \text{coh}(\Sigma B, \Sigma^{\text{Sph}} A, \varepsilon_C \text{Free} \kappa \circ \Sigma \text{Free}(\sigma_{\mathbb{X}} \tau^\dagger)) \\
&= \text{coh}(\Sigma B, \Sigma^{\text{Sph}} A, \varepsilon_C \circ \text{Free}(\kappa \circ \Sigma(\sigma_{\mathbb{X}} \tau^\dagger)))
\end{aligned}$$

so from the equality above and injectivity of the constructor coh , we get that $\Sigma B = \Sigma B'$, that $\Sigma^{\text{Sph}} A = \Sigma^{\text{Sph}} A'$, and that the corresponding morphisms agree. The suspension is injective on trees by injectivity of $\text{br}[\]$, so $B = B'$. The equality of morphisms implies that $\sigma_{\mathbb{X}} \tau^\dagger = \sigma_{\mathbb{X}} \tau'^\dagger$ by transposing along the adjunction $\text{Free} \vdash \text{Cell}$ and $\Sigma \vdash \Omega$. By structural induction on the n -cells, we may assume that $\sigma_{\mathbb{X}}$ is injective on the n -cells in the image of τ^\dagger from which we conclude that $\tau^\dagger = \tau'^\dagger$ and hence that $\tau = \tau'$. Finally by structural induction, we can see that Σ^{Cell} and Σ^{Sph} are monic, so $A = A'$ and $c = c'$.

To show that $\sigma_{\mathbb{X}}$ is surjective on n -cells, we proceed by structural induction on the cells. Since the function $\sigma_{\mathbb{X}, n, V}^{\text{ind}}$ is bijective, every indecomposable n -cell is in the image of $\sigma_{\mathbb{X}}$. Suppose therefore that $c \in X_n$ can be decomposed in the form

$$c = \alpha_{\mathbb{X}}(\text{coh}(B, A, \text{Free} \tau)) = \text{coh}(\Sigma B, \Sigma^{\text{Sph}} A, \varepsilon_C \circ \text{Free}(\kappa \circ \Sigma \tau)).$$

By structural induction, we may assume that for every position $p \in \text{Pos}_k(B)$, there exists a cell $\tau'(p) \in \text{Cell}_n(C_{\mathbb{X}})$ such that

$$\tau(p) = \sigma_{\mathbb{X}}(\tau'(p))$$

By injectivity of σ on cells of dimension at most k , those cells are unique and they can be assembled into a morphism $\tau' : \text{Pos}(B) \rightarrow \text{Cell}(C_{\mathbb{X}})$ such that $\tau = \sigma_{\mathbb{X}} \tau'$. Using this morphism and the calculations above, we see that

$$c = \alpha_{\mathbb{X}}(\text{coh}(B, A, \text{Free}(\sigma_{\mathbb{X}} \tau'))) = \sigma_{\mathbb{X}}(\text{coh}(B, A, \varepsilon \circ \text{Free} \tau')),$$

so $\sigma_{\mathbb{X}}$ is surjective on n -cells. □

Corollary 11. *The assignment $C \mapsto C_{\Omega K^{T^{**} C}}$ extends to a functor*

$$\Omega : \text{Comp}^{**} \rightarrow \text{Comp}$$

making the following two diagrams commute up to isomorphism:

$$\begin{array}{ccc}
\omega \text{Cat}^{**} & \xrightarrow{\Omega} & \omega \text{Cat} \\
K^{T^{**}} \uparrow & & \uparrow K^T \\
\text{Comp}^{**} & \xrightarrow{\Omega} & \text{Comp}
\end{array}
\qquad
\begin{array}{ccc}
\text{Comp}^{**} & \xrightarrow{\Omega} & \text{Comp} \\
\text{Cell}^{**} \downarrow & & \downarrow \text{Cell} \\
\text{Glob}^{**} & \xrightarrow{\Omega} & \text{Glob}
\end{array}$$

Proof. Since the comparison functor K^T is fully faithful, for every morphism of computads $\tau : C \rightarrow D$, we define $\Omega\tau$ to be the unique morphism making the following diagram commute:

$$\begin{array}{ccc} K^T \Omega C & \xrightarrow{\sim} & \Omega K^{T^{**}} C \\ K^T \Omega \tau \downarrow & & \downarrow \Omega K^{T^{**}} \tau \\ K^T \Omega D & \xrightarrow{\sim} & \Omega K^{T^{**}} D \end{array}$$

□

Remark 12. In general, the hom computad functor does not preserve finiteness. For example, consider the computad $\text{Free } \mathbb{D}^0$ with only one 0-dimensional generator x . The hom computad $\Omega(\text{Free } \mathbb{D}^0, x, x)$ has a countable number of 0-dimensional generators, corresponding to the identity of x and possible ways it can be composed.

5 Opposites

An important feature of ordinary category theory is the duality stemming from the existence of opposite categories. This feature extends to higher categories, where we may define opposites by reversing the direction of all cells in certain dimensions. In this section, we will define the opposite of a globular set, a computad, and an ω -category with respect to a set of dimensions $w \subseteq \mathbb{N}_{>0}$, using the same technique as in the previous section. We will then show that the formation of opposites in all those cases gives rise to an action of the Boolean group

$$G = \mathcal{P}(\mathbb{N}_{>0}) \cong \mathbb{Z}_2^{\mathbb{N}_{>0}}$$

of subsets of the positive integers with respect to symmetric difference. This group is clearly isomorphic to the group of functions $\mathbb{N}_{>0} \rightarrow \mathbb{Z}_2$ with pointwise multiplication, where each subset is identified with its indicator function. Abusing notation we will identify a subset w with its indicator function, and write $w(n)$ for the value of the indicator function at $n \in \mathbb{N}_{>0}$.

5.1 The opposite of a globular set

The group G acts on the category \mathbb{G} of globes by swapping the source and target inclusions. More precisely, an element $w \in G$ acts as the identity-on-objects functor

$$\text{op}_w : \mathbb{G} \rightarrow \mathbb{G}$$

given on the generating morphisms by

$$\text{op}_w(s_n) = \begin{cases} t_n & \text{if } n+1 \in w, \\ s_n & \text{if } n+1 \notin w, \end{cases} \quad \text{op}_w(t_n) = \begin{cases} s_n & \text{if } n+1 \in w, \\ t_n & \text{if } n+1 \notin w. \end{cases}$$

The functor op_\emptyset is clearly the identity functor. Moreover, for every pair of elements $w, w' \in G$, we can easily check that

$$\text{op}_w \text{op}_{w'} = \text{op}_{ww'}$$

so the assignment $w \mapsto \text{op}_w$ is a group homomorphism $G \rightarrow \text{Aut}(\mathbb{G})$. Since the group G is Abelian, this action extends to an action on the category Glob of globular sets by precomposition

$$\begin{aligned} \text{op}: G &\rightarrow \text{Aut}(\text{Glob}) \\ \text{op}_w(X) &= X \circ \text{op}_w. \end{aligned}$$

The opposite $\text{op}_w X$ of a globular set X therefore has the same cells as X , with the source and target of n -cells reversed for $n \in w$.

Since pasting diagrams are bipointed by their 0-source and 0-target inclusions, it will be useful to further extend this action to an action on bipointed globular sets

$$\text{op}: G \rightarrow \text{Aut}(\text{Glob}^{**})$$

by letting op_w take a bipointed globular set (X, x_-, x_+) to the opposite globular set $\text{op}_w X$ with the same basepoints when $1 \notin w$, and with the basepoints swapped otherwise.

Lemma 13. *For every $w \in G$, there exists a natural isomorphism*

$$\text{op}_w^\Sigma: \Sigma \circ \text{op}_{w-1} \Rightarrow \text{op}_w \circ \Sigma$$

where $w-1 \in G$ is the sequence defined by $(w-1)(n) = w(n+1)$. Moreover, $\text{op}_\emptyset^\Sigma$ is the identity natural transformation, and for every pair of elements $w, w' \in G$, the following diagram commutes:

$$\begin{array}{ccc} \Sigma \text{op}_{w-1} \text{op}_{w'-1} & \xrightarrow{\text{op}_w^\Sigma \text{op}_{w'-1}} & \text{op}_w \Sigma \text{op}_{w'-1} & \xrightarrow{\text{op}_w \text{op}_{w'}^\Sigma} & \text{op}_w \text{op}_{w'} \Sigma \\ \parallel & & & & \parallel \\ \Sigma \text{op}_{ww'-1} & \xrightarrow{\text{op}_{ww'}^\Sigma} & & & \text{op}_{ww'} \Sigma \end{array}$$

Proof. For every globular set X , the bipointed globular sets $\Sigma \text{op}_{w-1} X$ and $\text{op}_w \Sigma X$ have the same sets of cells. Moreover, the source and target of an n -cell in both of them agree when $n > 2$: they are given by the target and source functions of X respectively when $n \in w$, and they are given by the source and target functions of X when $n \notin w$. The source and target of a 1-cell in the first one are given by v^- and v^+ respectively, while in the latter it is given by those when $1 \notin w$, and by v^+ and v^- when $1 \in w$. Therefore, we may define an isomorphism of globular sets

$$\text{op}_{w,X}^\Sigma: \Sigma \text{op}_{w-1} X \rightarrow \text{op}_w \Sigma X$$

to be the identity on positive-dimensional cells, and to be given on 0-cells by

$$\text{op}_{w,X}^\Sigma(v_\pm) = \begin{cases} \text{op}_{w,X}^\Sigma(v_\mp), & \text{if } 1 \in w, \\ \text{op}_{w,X}^\Sigma(v_\pm), & \text{if } 1 \notin w. \end{cases}$$

Since op_w reverses the basepoints if and only if $1 \in w$, we see that this is a morphism of bipointed globular sets. Naturality of these morphisms follows easily by the fact that it is the identity of positive-dimensional cells. Finally, the claimed diagram commutes for $w, w' \in G$: both morphisms are identity on positive-dimensional cells, they are the identity on 0-cells when $1 \in w \cap w'$ or $1 \notin w \cup w'$, and they swap the two 0-cells otherwise. \square

Lemma 14. *For every $w \in G$ and $n \in \mathbb{N}$, there exists a natural isomorphism*

$$\text{op}_w^\vee: \bigvee_{i=1}^n \circ \text{swap}_{w(1)} \circ (\text{op}_w)^n \Rightarrow \text{op}_w \circ \bigvee_{i=1}^n$$

where swap_0 is the identity of $(\text{Glob}^{**})^n$, while swap_1 is the automorphism

$$\text{swap}_1(X_1, \dots, X_n) = (X_n, \dots, X_1).$$

Moreover, op_\emptyset^\vee is the identity natural transformation, and for every pair of elements $w, w' \in G$, the following diagram commutes:

$$\begin{array}{ccc} \bigvee \circ (\text{op}_{ww'})^n \circ \text{swap}_{ww'(1)} & \xrightarrow{\text{op}_{ww'}^\vee} & \text{op}_{ww'} \circ \bigvee \\ \parallel & & \parallel \\ \bigvee \circ (\text{op}_w)^n \circ (\text{op}_{w'})^n \circ \text{swap}_{w(1)} \circ \text{swap}_{w'(1)} & & \\ \parallel & & \\ \bigvee \circ (\text{op}_w)^n \circ \text{swap}_{w(1)} \circ (\text{op}_{w'})^n \circ \text{swap}_{w'(1)} & & \\ \downarrow \text{op}_w^\vee & & \\ \text{op}_w \circ \bigvee \circ (\text{op}_{w'})^n \circ \text{swap}_{w'(1)} & \xrightarrow{\text{op}_w(\text{op}_{w'}^\vee)} & \text{op}_w \text{op}'_w \circ \bigvee \end{array}$$

Proof. Fix $n \in \mathbb{N}$ and $w \in G$ and let X_1, \dots, X_n be bipointed globular sets and suppose first that $1 \notin w$, so that the basepoints of X_i and $\text{op}_w X_i$ agree. The functor op_w on globular sets preserves \mathbb{D}^0 , and it preserves colimits, being an equivalence of categories. Therefore, there exists a natural isomorphism of globular sets

$$\text{op}_w^\vee: \bigvee_{i=1}^n (\text{op}_w X_i) \rightarrow \text{op}_w \left(\bigvee_{i=1}^n X_i \right),$$

that can be easily seen to preserve the basepoints. Moreover, since op_w preserves the cells of a globular set, and colimits of globular sets are computed pointwise, we may take op_w^\vee to be the identity.

Suppose now that $1 \in w$, so that the functor op_w swaps the basepoints. Using that op_w preserves colimits and \mathbb{D}^0 , we see that $\text{op}_w(\bigvee_{i=1}^n X_i)$ is the colimit of the following diagram.

$$\begin{array}{ccccc} \text{op}_w X_1 & & \dots & & \text{op}_w X_n \\ & \swarrow & & \swarrow & \\ & x_{1,+} & & x_{n,-} & \\ & \mathbb{D}^0 & & \mathbb{D}^0 & \\ & \searrow & & \searrow & \\ & x_{2,-} & & x_{n-1,+} & \end{array}$$

On the other hand, $\bigvee_{i=n}^1 \text{op}_w X_i$ is the colimit of the following diagram:

$$\begin{array}{ccccc}
 \text{op}_w X_n & & \dots & & \text{op}_w X_1 \\
 & \swarrow & & \nwarrow & \\
 & x_{n,-} & & x_{2,-} & \\
 & & \mathbb{D}^0 & & \mathbb{D}^0 \\
 & \searrow & & \swarrow & \\
 & x_{n-1,+} & & x_{1,+} &
 \end{array}$$

By symmetry of pushouts, we get a natural isomorphism of globular sets

$$\text{op}_w^\vee : \bigvee_{i=n}^1 (\text{op}_w X_i) \rightarrow \text{op}_w \left(\bigvee_{i=1}^n X_i \right)$$

that can be easily seen to preserve the basepoints. Since colimits are computed object-wisely, this isomorphism is given level-wisely by the symmetry of pushouts.

Knowing how those isomorphisms are defined pointwise, we can easily deduce that the claimed diagram commutes for every pair $w, w' \in G$. If $1 \notin w \cup w'$, then both sides of the diagram are identities. If $1 \in w \cap w'$ again both are identities, since the symmetry of the pushout squares to the identity. Finally, when $1 \in ww'$, then both sides are given by the symmetry of the pushout, so they agree. \square

Using those lemmas, we can deduce that pasting diagrams are closed under the formation of opposites: we define recursively on the Batanin tree B for every $w \in G$ the w -opposite Batanin tree $\text{op}_w B$ by the formula

$$\text{op}_w(\text{br}[B_1, \dots, B_n]) = \text{br}_{\text{swap}_{w(1)}}[\text{op}_{w-1} B_1, \dots, \text{op}_{w-1} B_n],$$

where swap_0 is the identity of the set of lists, while swap_1 reverses a list

$$\text{swap}_1[B_1, \dots, B_n] = [B_n, \dots, B_1].$$

The opposite tree realizes the opposite pasting diagram, in the sense that there exists an isomorphism of bipointed globular sets

$$\text{op}_w^B : \text{Pos}(\text{op}_w B) \rightarrow \text{op}_w(\text{Pos}(B)).$$

We can define this isomorphism recursively on $B = \text{br}[B_1, \dots, B_n]$ to be the following composite

$$\begin{aligned}
 \text{Pos}(\text{op}_w B) &= \bigvee_{i=1}^n \text{swap}_{w(1)}(\Sigma \text{Pos}(\text{op}_{w-1} B_i)) \\
 &\xrightarrow{\bigvee \text{swap}_{w(1)} \Sigma \text{op}_{w-1}^{B_i}} \bigvee_{i=1}^n \text{swap}_{w(1)}(\Sigma \text{op}_{w-1} \text{Pos}(B_i)) \\
 &\xrightarrow{\bigvee \text{swap}_{w(1)} \text{op}_w^\Sigma} \bigvee_{i=1}^n \text{swap}_{w(1)}(\text{op}_w \Sigma \text{Pos}(B_i)) \\
 &\xrightarrow{\text{op}_w^\vee} \text{op}_w \left(\bigvee_{i=1}^n \Sigma \text{Pos}(B_i) \right) = \text{op}_w(\text{Pos}(B)).
 \end{aligned}$$

Lemma 15. *The isomorphism op_0^B is the identity for every tree B , and for any $w, w' \in G$, the following diagram of isomorphisms commutes:*

$$\begin{array}{ccc}
\text{Pos}(\text{op}_w \text{op}_{w'} B) & \xrightarrow{\text{op}_w^{\text{op}_{w'} B}} & \text{op}_w(\text{Pos}(\text{op}_{w'} B)) & \xrightarrow{\text{op}_w(\text{op}_{w'}^B)} & \text{op}_w \text{op}_{w'} \text{Pos}(B) \\
\parallel & & & & \parallel \\
\text{Pos}(\text{op}_{ww'} B) & \xrightarrow{\text{op}_{ww'}^B} & & & \text{op}_{ww'} \text{Pos}(B)
\end{array}$$

Proof. This lemma is an easy induction on B , using naturality of the isomorphisms in Lemmas 13 and 14, and of the commuting diagrams there. \square

Lemma 16. *For every $w \in G$, $k \in \mathbb{N}$ and Batanin tree B ,*

$$\text{op}_w \partial_k = \partial_k \text{op}_w .$$

Moreover, the following equations hold

$k + 1 \in w$	$k + 1 \notin w$
$\text{op}_w(t_k^B) \circ \text{op}_w^{\partial_k B} = \text{op}_w^B \circ s_k^{\text{op}_w B}$	$\text{op}_w(s_k^B) \circ \text{op}_w^{\partial_k B} = \text{op}_w^B \circ s_k^{\text{op}_w B}$
$\text{op}_w(s_k^B) \circ \text{op}_w^{\partial_k B} = \text{op}_w^B \circ t_k^{\text{op}_w B}$	$\text{op}_w(t_k^B) \circ \text{op}_w^{\partial_k B} = \text{op}_w^B \circ t_k^{\text{op}_w B}$

Proof. We proceed by induction on k . For $k = 0$ both $\text{op}_w \partial_k B$ and $\partial_k \text{op}_w B$ are equal to the disk D_0 , and the equations state that op_w^B preserves the basepoints. Suppose therefore that the result is true for some $k \in \mathbb{N}$ to prove that it also holds for $k + 1$. Letting $B = \text{br}[B_1, \dots, B_n]$, we see that

$$\begin{aligned}
\text{op}_w \partial_{k+1} B &= \text{op}_w(\text{br}[\partial_k B_1, \dots, \partial_k B_n]) \\
&= \text{br}(\text{swap}_{w(1)}[\text{op}_{w-1} \partial_k B_1, \dots, \text{op}_{w-1} \partial_k B_n]) \\
&= \text{br}(\text{swap}_{w(1)}[\partial_k \text{op}_{w-1} B_1, \dots, \partial_k \text{op}_{w-1} B_n]) \\
&= \partial_{k+1} \text{br}(\text{swap}_{w(1)}[\text{op}_{w-1} B_1, \dots, \text{op}_{w-1} B_n]) \\
&= \partial_{k+1} \text{op}_w B
\end{aligned}$$

by the inductive hypothesis.

We will prove the first equation in the case that $k + 1 \in w$ and $1 \in w$. The other equation and the rest of the cases follow by the same argument. By the inductive hypothesis, we may assume that for $1 \leq i \leq n$, the following square

commutes

$$\begin{array}{ccc}
\text{Pos}(\text{op}_{w-1} B_i) & \xrightarrow{\text{op}_{w-1}^{B_i}} & \text{op}_{w-1}(\text{Pos}(B_i)) \\
\uparrow \text{op}_{w-1}^{B_i} s_{k-1} & & \uparrow \text{op}_{w-1}(t_{k-1}^{B_i}) \\
\text{Pos}(\partial_{k-1} \text{op}_{w-1} B_i) & & \text{Pos}(\partial_{k-1} B_i) \\
\parallel & & \parallel \\
\text{Pos}(\text{op}_{w-1} \partial_{k-1} B_i) & \xrightarrow[\text{op}_{w-1}]{\partial_{k-1} B_i} & \text{op}_{w-1} \text{Pos}(\partial_{k-1} B_i)
\end{array}$$

Applying the suspension functor and then the wedge sum from n to 1, we get that the left square below commutes. Naturality of the isomorphisms in Lemmas 13 and 14 then imply that the right square below also commutes.

$$\begin{array}{ccccc}
\bigvee \Sigma \text{Pos}(\text{op}_{w-1} B_i) & \longrightarrow & \bigvee \Sigma \text{op}_{w-1} \text{Pos}(B_i) & \longrightarrow & \text{op}_w \bigvee \Sigma \text{Pos}(B_i) \\
\uparrow \bigvee \Sigma s_{k-1}^{\text{op}_{w-1} B_i} & & \uparrow & & \uparrow \\
\bigvee \Sigma \text{Pos}(\partial_{k-1} \text{op}_{w-1} B_i) & & \bigvee \Sigma \text{op}_{w-1}(t_{k-1}^{B_i}) & & \text{op}_w \bigvee \Sigma t_{k-1}^{B_i} \\
\parallel & & \parallel & & \parallel \\
\bigvee \Sigma \text{Pos}(\text{op}_{w-1} \partial_{k-1} B_i) & \rightarrow & \bigvee \Sigma \text{op}_{w-1} \text{Pos}(\partial_{k-1} B_i) & \rightarrow & \text{op}_w \bigvee \Sigma \text{Pos}(\partial_{k-1} B_i)
\end{array}$$

The outer part of the diagram though is precisely the square:

$$\begin{array}{ccc}
\text{Pos}(\text{op}_w B) & \xrightarrow{\text{op}_w^B} & \text{op}_w(\text{Pos}(B)) \\
\uparrow s_k^{\text{op}_w B} & & \uparrow \text{op}_w(t_k^B) \\
\text{Pos}(\partial_k \text{op}_w B) & & \text{Pos}(\partial_k B) \\
\parallel & & \parallel \\
\text{Pos}(\text{op}_w \partial_k B) & \xrightarrow[\text{op}_w]{\partial_k B} & \text{op}_w \text{Pos}(\partial_k B)
\end{array}$$

whose commutativity amounts to the first equation. \square

Remark 17. Batanin trees famillially represent the free strict ω -category monad T^{str} on the category of globular sets, in the sense that

$$(T^{\text{str}} X)_n = \coprod_{\dim B \leq n} \text{Glob}(\text{Pos}(B), X)$$

as shown by Leinster [20]. Compatibility of the opposites with Batanin trees allows us to define an invertible morphism of monads $\text{op}_w^{\text{str}} : T^{\text{str}} \text{op}_w \Rightarrow \text{op}_w T^{\text{str}}$ by the formula

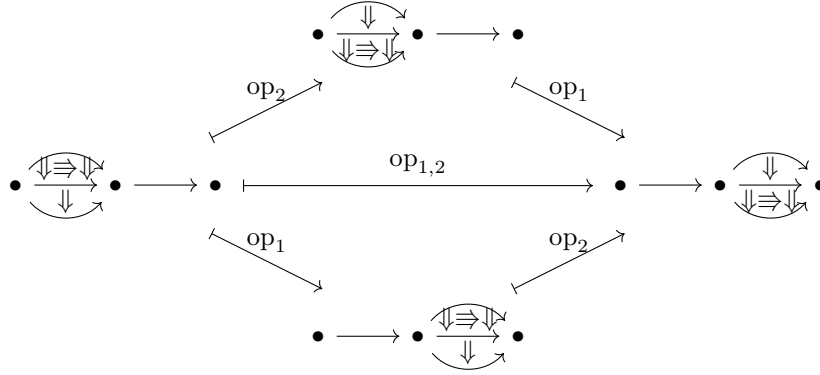
$$\text{op}_{w,X}^{\text{str}}(B, f : \text{Pos}(B) \rightarrow \text{op}_w X) = (\text{op}_w B, \text{op}_w f \circ \text{op}_w^B)$$

hence providing an alternative definition of the opposites of a strict ω -category.

Example 18. The disk trees D_n defined in Example 4 are invariant under the action of opposites. In other words for every $w \in G$, we have $\text{op}_w D_n = D_n$. However, in general, the isomorphism $\text{op}_w^{D_n}$ is not the identity. The family of trees $B_{n,k,m}$ of Example 5 satisfies the following equality with respect to the opposites:

$$\text{op}_w(B_{n,k,m}) = \begin{cases} B_{m,k,n} & \text{if } k+1 \in w \\ B_{n,k,m} & \text{otherwise.} \end{cases}$$

Example 19. As a richer example, consider the tree $B = \text{br}[\text{br}[\text{br}[\text{br}[]], \text{br}[]], \text{br}[]]$. We illustrate the various opposites of this tree on the corresponding globular pasting diagram with the following figure:



5.2 The opposite of a computad

The opposite of a computad is defined similarly to the opposite of a globular set by swapping the source and target of its generators. To define this action, we fix an element $w \in G$, that will be omitted from the notation, and define recursively on the dimension $n \in \mathbb{N}$, an endofunctor and two natural transformation

$$\begin{aligned} \text{op} &: \text{Comp}_n \rightarrow \text{Comp}_n \\ \text{op}^{\text{Cell}} &: \text{Cell}_n \Rightarrow \text{Cell}_n \text{ op} \\ \text{op}^{\text{Sph}} &: \text{Sph}_n \Rightarrow \text{Sph}_n \text{ op} \end{aligned}$$

satisfying the following properties:

(OP1) forming opposites commutes with the forgetful functors, and the inclusion of globular sets into computads

$$\begin{array}{ccc} \text{Comp}_{n+1} & \xrightarrow{\text{op}} & \text{Comp}_{n+1} \\ u_{n+1} \downarrow & & \downarrow u_{n+1} \\ \text{Comp}_n & \xrightarrow{\text{op}} & \text{Comp}_n \end{array} \quad \begin{array}{ccc} \text{Glob} & \xrightarrow{\text{op}} & \text{Glob} \\ \text{Free}_n \downarrow & & \downarrow \text{Free}_n \\ \text{Comp}_n & \xrightarrow{\text{op}} & \text{Comp}_n \end{array}$$

(OP2) the natural transformations are compatible with the boundary natural transformation

$$\begin{array}{ccc}
\text{Cell}_{n+1} & \xrightarrow{\text{op}^{\text{Cell}}} & \text{Cell}_{n+1} \text{ op} \\
\text{bdry}_{n+1} \Downarrow & & \Downarrow \text{bdry}_{n+1} \text{ op} \\
\text{Sph}_n u_{n+1} & \xrightarrow{\text{op}^{\text{Sph}}_{u_{n+1}}} & \text{Sph}_n \text{ op } u_{n+1} = \text{Sph}_n u_{n+1} \text{ op}
\end{array}$$

(OP3) the natural transformation op^{Sph} swaps the two cells of a sphere when $n+1 \in w$ and leaves them unchanged otherwise, in the sense that the following diagrams commute for $i = 1, 2$

$$\begin{array}{ccc}
\text{Sph}_n & \xrightarrow{\text{op}^{\text{Sph}}} & \text{Sph}_n \text{ op} \\
\text{pr}_i \Downarrow & n+1 \notin w & \Downarrow \text{pr}_i \text{ op} \\
\text{Cell}_n & \xrightarrow{\text{op}^{\text{Cell}}} & \text{Cell}_n \text{ op}
\end{array}
\qquad
\begin{array}{ccc}
\text{Sph}_n & \xrightarrow{\text{op}^{\text{Sph}}} & \text{Sph}_n \text{ op} \\
\text{pr}_i \Downarrow & n+1 \in w & \Downarrow \text{pr}_{2-i} \text{ op} \\
\text{Cell}_n & \xrightarrow{\text{op}^{\text{Cell}}} & \text{Cell}_n \text{ op}
\end{array}$$

(OP4) the natural transformation op^{Cell} preserves generators, in that for every globular set X and $x \in X_n$, we have that

$$\text{op}_{\text{Free } X}^{\text{Cell}}(\text{var } x) = \text{var } x.$$

(OP5) the natural transformation op^{Sph} preserves fullness, in that for every full n -sphere A of $\text{Free Pos}(B)$, the n -sphere

$$A' = \text{Sph}_n \text{Free}_n(\text{op}^B)^{-1}(\text{op}^{\text{Sph}}(A))$$

of $\text{Free Pos}(\text{op } B)$ is also full.

As a base case, we define op and op^{Sph} to be the identities of Comp_{-1} and Sph_{-1} respectively. Let therefore $n \in \mathbb{N}$ and suppose inductively that data as above has been defined for all natural numbers less than n , satisfying the given properties.

Computads. First we will define the action of op on all n -computads. Let $C = (C_{n-1}, V_n^C, \phi_n^C)$ be an n -computad. The opposite computad $\text{op } C$ consists of the opposite computad $\text{op } C_{n-1}$, the same set of generators V_n^C , and the attaching function

$$\phi_n^{\text{op } C} : V_n^C \xrightarrow{\phi_n^C} \text{Sph}_{n-1} C_{n-1} \xrightarrow{\text{op}^{\text{Sph}}} \text{Sph}_{n-1}(\text{op } C_{n-1}).$$

By Properties (OP3) and (OP4), we can easily deduce that op commutes with the inclusion Free_n on objects, while it clearly commutes with the forgetful functors u_n by definition.

Cells and morphisms. We will then define op on morphisms σ of n -computads of target C , together with the component of the natural transformation op^{Cell} at C mutually recursively. For a generator $v \in V_n^C$, we let

$$\text{op}^{\text{Cell}}(\text{var } v) = \text{var } v,$$

and we observe that

$$\text{bdry}_n \text{op}^{\text{Cell}}(\text{var } v) = \text{op}^{\text{Sph}} \text{bdry}_n(\text{var } v).$$

Given a coherence cell $c = \text{coh}(B, A, \tau)$ of C , we may assume that recursively that $\text{op}(\tau)$ has been defined, and let

$$\begin{aligned} A' &= \text{Sph}_{n-1} \text{Free}_{n-1}(\text{op}^B)^{-1}(\text{op}^{\text{Sph}}(A)) \\ \text{op}^{\text{Cell}}(c) &= \text{coh}(\text{op } B, A', \text{op}(\tau) \circ \text{Free}_n(\text{op}^B)) \end{aligned}$$

We then observe again that the boundary of this cell is given by

$$\begin{aligned} \text{bdry}_n \text{op}^{\text{Cell}}(c) &= \text{Sph}_{n-1}(\text{op } \tau_{n-1})(\text{op}^{\text{Sph}} A) \\ &= \text{op}^{\text{Sph}}(\text{Sph}_{n-1}(\tau_{n-1})(A)) \\ &= \text{op}^{\text{Sph}} \text{bdry}_n(c). \end{aligned}$$

Finally, for a morphism $\sigma = (\sigma_{n-1}, \sigma_V) : D \rightarrow C$, we define assume that op^{Cell} has been defined on cells of the form $\sigma_V(v)$ for $v \in V_n^D$ and define

$$\text{op}(\sigma) = (\text{op } \sigma_{n-1}, \text{op}^{\text{Cell}} \circ \sigma_V) : \text{op } D \rightarrow \text{op } C$$

This is a well-defined morphism of computads by the observation on the boundary of the cells $\text{op}^{\text{Cell}}(c)$, i.e. by Properties (OP2).

It follows immediately from the definition that op commutes with the forgetful functor u_n on morphisms as well. Using that op^{Cell} preserves generators, we can also deduce that op commutes with the inclusion Free_n on morphisms as well. Therefore, we have shown Properties (OP1), (OP2) and (OP4) so far.

Naturality. We will now show that op is a functor and that op^{Cell} is natural. For that, we fix a morphism of n -computads $\sigma : C \rightarrow D$, and we proceed recursively to show that the following square commutes

$$\begin{array}{ccc} \text{Cell}_n C & \xrightarrow{\text{Cell}_n \sigma} & \text{Cell}_n D \\ \text{op}^{\text{Cell}} \downarrow & & \downarrow \text{op}^{\text{Cell}} \\ \text{Cell}_n \text{op } C & \xrightarrow{\text{Cell}_n \text{op } \sigma} & \text{Cell}_n \text{op } D \end{array}$$

and that for all morphism $\tau : E \rightarrow C$,

$$\text{op } \sigma \circ \text{op } \tau = \text{op}(\sigma \circ \tau).$$

By definition of $\text{op } \sigma$, the square above commutes when restricted to generators. Moreover, for a coherence cell $c = \text{coh}(B, A, \tau)$, we see that

$$\begin{aligned}
\text{op}^{\text{Cell}} \circ \text{Cell}_n(\sigma)(c) &= \text{op}^{\text{Cell}}(\text{coh}(B, A, \sigma \circ \tau)) \\
&= \text{coh}(B, A', \text{op}(\sigma \circ \tau) \circ \text{Free}(\text{op}^B)) \\
&= \text{coh}(B, A', \text{op}(\sigma) \circ \text{op}(\tau) \circ \text{Free}(\text{op}^B)) \\
&= \text{Cell}_n(\text{op } \sigma)(\text{coh}(B, A', \text{op}(\tau) \circ \text{Free}(\text{op}^B))) \\
&= \text{Cell}_n(\text{op } \sigma) \circ \text{op}^{\text{Cell}}(\text{coh}(B, A, \tau))
\end{aligned}$$

where A' is defined as above. Given arbitrary $\tau: E \rightarrow C$, we may assume that the square commutes when restricted to the image of τ_V . By the inductive hypothesis, op preserves composition of morphisms of $(n-1)$ -computads. Hence it suffices to show the equality above for the generators of \mathbf{E} . We recall the definition of the composition of morphisms of n -computads given in [11, Section 3.1]:

$$(\sigma_{n-1}, \sigma_v) \circ (\tau_{n-1}, \tau_V) = (\sigma_{n-1} \circ \tau_{n-1}, \text{Cell}_n(\sigma) \circ \tau_v)$$

Using this definition, we have:

$$\begin{aligned}
(\text{op}(\sigma \circ \tau))_V &= \text{op}^{\text{Cell}} \circ \text{Cell}_n(\sigma) \circ \tau_V \\
&= \text{Cell}_n(\text{op } \sigma) \circ \text{op}_n^{\text{Cell}} \circ \tau_V \\
&= (\text{op}(\sigma) \circ \text{op}(\tau))_V.
\end{aligned}$$

Therefore, op is a functor and op^{Cell} is natural.

Spheres. The natural transformation op^{Sph} is completely determined by Property (OP3). Indeed, for an n -computad C and for a sphere $(a, b) \in \text{Sph}_n C$, we are forced to define

$$\text{op}^{\text{Sph}}(a, b) = \begin{cases} (\text{op}^{\text{Cell}} b, \text{op}^{\text{Cell}} a), & \text{if } n+1 \in w \\ (\text{op}^{\text{Cell}} a, \text{op}^{\text{Cell}} b), & \text{if } n+1 \notin w \end{cases}$$

Property (OP2) shows us that those $\text{op}^{\text{Cell}} a$ and $\text{op}^{\text{Cell}} b$ have the same source and target, so that this assignment is well-defined. It is clearly natural by naturality of op^{Cell} .

Fullness. To finish the recursive definition, it remains to show that for every Batanin tree B and every n -sphere $A = (a, b)$ of $\text{Free}_n \text{Pos}(B)$, the n -sphere

$$A' = \text{Sph}_n \text{Free}_n(\text{op}^B)^{-1}(\text{op}^{\text{Sph}}(A))$$

of $\text{Free}_n \text{Pos}(\text{op } B)$ is also full. We will show that in the case that $n+1 \in w$, the other case being similar. By assumption, we may write

$$a = \text{Cell}_n \text{Free}_n(s_n^B)(a_0) \quad b = \text{Cell}_n \text{Free}_n(t_n^B)(b_0).$$

For n -cells a_0, b_0 of $\text{Free}_n \text{Pos}(\partial_n B)$ whose support contains all positions of $\partial_n B$. Then we have that $A' = (a', b')$ where

$$\begin{aligned} a' &= \text{Cell}_n \text{Free}_n((\text{op}^B)^{-1} \circ t_n^B)(\text{op}^{\text{Cell}} b_0) \\ b' &= \text{Cell}_n \text{Free}_n((\text{op}^B)^{-1} \circ t_n^B)(\text{op}^{\text{Cell}} a_0) \end{aligned}$$

By Lemma 16, we can rewrite those cells as

$$\begin{aligned} a' &= \text{Cell}_n \text{Free}_n(s_n^{\text{op}^B})(\text{Cell}_n \text{Free}_n(\text{op}^{\partial_n B})^{-1}(\text{op}^{\text{Cell}} b_0)) \\ b' &= \text{Cell}_n \text{Free}_n(t_n^{\text{op}^B})(\text{Cell}_n \text{Free}_n(\text{op}^{\partial_n B})^{-1}(\text{op}^{\text{Cell}} a_0)). \end{aligned}$$

Using the definition of the support and that op^{Cell} preserves generators, we may show recursively that

$$\text{supp}(\text{op}^{\text{Cell}}(c)) = \text{supp}(c)$$

for every cell c . Moreover, isomorphisms of computads induce bijections on the support of cells, so the support of the cells

$$\begin{aligned} &\text{Cell}_n \text{Free}_n(\text{op}^{\partial_n B})^{-1}(\text{op}^{\text{Cell}} b_0) \\ &\text{Cell}_n \text{Free}_n(\text{op}^{\partial_n B})^{-1}(\text{op}^{\text{Cell}} a_0) \end{aligned}$$

must contain all positions of $\partial_n \text{op} B$. Therefore, A' is full.

Lemma 20. *For every $n \in \mathbb{N}$, the endofunctor op_\emptyset on n -computads is the identity, and so are the natural transformations $\text{op}_\emptyset^{\text{Cell}}$ and $\text{op}_\emptyset^{\text{Sph}}$. Moreover, for any pair of elements $w, w' \in G$,*

$$\text{op}_w \text{op}_{w'} = \text{op}_{ww'}$$

and the following diagrams commute.

$$\begin{array}{ccccc} \text{Cell}_n & \xrightarrow{\text{op}_{w'}^{\text{Cell}}} & \text{Cell}_n \text{op}_{w'} & \xrightarrow{\text{op}_w^{\text{Cell}} \text{op}_{w'}} & \text{Cell}_n \text{op}_w \text{op}_{w'} \\ \parallel & & & & \parallel \\ \text{Cell}_n & \xrightarrow{\text{op}_{ww'}^{\text{Cell}}} & & & \text{Cell}_n \text{op}_{ww'} \\ \\ \text{Sph}_n & \xrightarrow{\text{op}_{w'}^{\text{Sph}}} & \text{Sph}_n \text{op}_{w'} & \xrightarrow{\text{op}_w^{\text{Sph}} \text{op}_{w'}} & \text{Sph}_n \text{op}_w \text{op}_{w'} \\ \parallel & & & & \parallel \\ \text{Sph}_n & \xrightarrow{\text{op}_{ww'}^{\text{Sph}}} & & & \text{Sph}_n \text{op}_{ww'} \end{array}$$

In particular, op_w , $\text{op}_w^{\text{Cell}}$ and op_w^{Sph} are invertible with inverses op_w , $\text{op}_w^{\text{Cell}} \text{op}_w$ and $\text{op}_w^{\text{Sph}} \text{op}_w$ respectively.

Proof. We proceed inductively on $n \in \mathbb{N}$, since the result holds trivially for $n = -1$. Since op_\emptyset and $\text{op}_\emptyset^{\text{Sph}}$ are identities for $(n-1)$ -computads, we see that

$$\text{op}_\emptyset C = C$$

for every n -computad C . Using Lemma 16, we can then show mutually recursively for an n -computad C that

$$\text{op}_\emptyset \sigma = \sigma \qquad \text{op}_\emptyset^{\text{Cell}} c = c$$

for every morphism $\sigma: D \rightarrow C$ and every n -cell c of C . Using then that $\text{op}_\emptyset^{\text{Sph}}$ is defined using $\text{op}_\emptyset^{\text{Cell}}$, we see that $\text{op}_\emptyset^{\text{Sph}}$ must be the identity as well.

Let now $w, w' \in G$ and $C = (C_{n-1}, V_n^C, \phi_n^C)$ an n -computad. Then the n -computad $\text{op}_w \text{op}_{w'} C$ consists of the $(n-1)$ -computad

$$\text{op}_w \text{op}_{w'} C_{n-1} = \text{op}_{ww'} C_{n-1},$$

the same set of generators, and the attaching function

$$\phi_n^{\text{op}_w \text{op}_{w'} C} = \text{op}_{w, \text{op}_{w'} C}^{\text{Sph}} \circ \text{op}_{w', C}^{\text{Sph}} \circ \phi_n^C = \text{op}_{ww', C}^{\text{Sph}} \circ \phi_n^C = \phi_n^{\text{op}_{ww'} C}.$$

Hence, $\text{op}_w \text{op}_{w'}$ and $\text{op}_{ww'}$ agree on n -computads. Fixing a computad C , we can show that they also agree on morphisms with target C mutually inductively to recursively to showing that the claimed diagram for $\text{op}_w^{\text{Cell}}$ commutes. The commutative diagram from op_w^{Sph} then follows from the one for $\text{op}_w^{\text{Cell}}$. \square

Having defined the opposite of an n -computad for every $n \in \mathbb{N}$, in a way that is compatible with the forgetful functors u_n , we get for every $w \in G$, a functor

$$\text{op}: \text{Comp} \rightarrow \text{Comp}$$

sending a computad $C = (C_n)_{n \in \mathbb{N}}$ to the computad

$$\text{op} C = (\text{op} C_n)_{n \in \mathbb{N}}$$

and acts similarly on morphisms. Property (OP1) shows that op is compatible with the inclusion functors Free in that

$$\begin{array}{ccc} \text{Comp} & \xrightarrow{\text{op}} & \text{Comp} \\ \text{Free} \uparrow & & \uparrow \text{Free} \\ \text{Glob} & \xrightarrow{\text{op}} & \text{Glob} \end{array}$$

commutes. Moreover, combining Properties (OP2) and (OP3), we see that the natural transformations op^{Cell} give rise to a natural transformation

$$\text{op}^{\text{Cell}}: \text{op Cell} \Rightarrow \text{Cell op}.$$

The following lemma is an easy consequence of Lemma 20.

Lemma 21. *The functor $\text{op}_\emptyset: \text{Comp} \rightarrow \text{Comp}$ is the identity functor, and the natural transformations $\text{op}_\emptyset^{\text{Cell}}$ is the identity of Cell . Moreover, for any pair $w, w' \in G$,*

$$\text{op}_w \text{op}_{w'} = \text{op}_{ww'}$$

and the following diagrams commute.

$$\begin{array}{ccc}
\text{op}_w \text{op}_{w'} \text{Cell} \xrightarrow{\text{op}_w \text{op}_{w'}^{\text{Cell}}} & \text{op}_w \text{Cell op}_{w'} \xrightarrow{\text{op}_w^{\text{Cell}} \text{op}_{w'}} & \text{Cell op}_w \text{op}_{w'} \\
\parallel & & \parallel \\
\text{op}_{ww'} \text{Cell} \xrightarrow{\text{op}_{ww'}^{\text{Cell}}} & & \text{Cell op}_{ww'}
\end{array}$$

In particular, each op_w is invertible with inverse itself, and $\text{op}_w^{\text{Cell}}$ is invertible with inverse $\text{op}_w \text{op}_w^{\text{Cell}}$.

Example 22. The family of cells $\text{comp}_{n,k,m} \in T\text{Pos}(B_{n,k,m})$ defined in Section 3.1 satisfies the following identities under the action of composites:

$$\text{op}_w^{\text{Cell}}(\text{comp}_{n,k,m}) = \begin{cases} \text{comp}_{m,k,n} & \text{if } k+1 \in w \\ \text{comp}_{n,k,m} & \text{otherwise.} \end{cases}$$

We defined $\text{comp}_{n,k,m}$ with $m > n$ by analogy, informally relying on the reader's ability to construct it from the case where $n < m$. The construction of opposites give us a way to make this argument formal.

5.3 The opposite of an ω -category.

So far, we have defined the opposite of a globular set, a pasting diagram and a computad. To extend those definitions and define the opposite of an ω -category, we consider the mate

$$\begin{aligned}
\text{op}^T : T \text{op}_w &\Rightarrow \text{op} T \\
\text{op}^T &= (\text{op}^{\text{Cell}} \text{Free})^{-1}
\end{aligned}$$

of the natural transformation $\text{op}^{\text{Cell}} \text{Free}$ under the adjunction $\text{op} \dashv \text{op}$. We will show that the functor $\text{op} : \text{Glob} \rightarrow \text{Glob}$ together with the natural transformation op^T is a morphism of monads from T to T . This amounts to commutativity of the following two diagrams

$$\begin{array}{ccc}
T \text{op} \xrightarrow{\text{op}^T} \text{op} T & & T T \text{op} \xrightarrow{T \text{op}^T} T \text{op} T \xrightarrow{\text{op}^T T} \text{op} T T \\
\eta \text{op} \swarrow & & \mu \text{op} \downarrow \\
& \text{op} & T \text{op} \xrightarrow{\text{op}^T} \text{op} T \\
& \searrow \text{op} \eta & \downarrow \text{op} \mu
\end{array}$$

The left one is the assertion that op^{Cell} preserves generators, which we have already shown. The right one is obtained from the following diagram by whiskering on the right with Free , and then replacing $\text{op}^{\text{Cell}} \text{Free}$ with its inverse.

$$\begin{array}{ccc}
\text{op} T \text{Cell} \xrightarrow{\text{op}^{\text{Cell}} \text{Free Cell}} \text{Cell op Free Cell} & \xlongequal{\quad} & T \text{op Cell} \xrightarrow{T \text{op}^{\text{Cell}}} T \text{Cell op} \\
\text{op Cell } \varepsilon \downarrow & & \downarrow \text{Cell } \varepsilon \text{ op} \\
\text{op Cell} \xrightarrow{\text{op}^{\text{Cell}}} \text{Cell op} & \xrightarrow{\text{Cell op } \varepsilon} & \text{Cell op}
\end{array}$$

The left square commutes by naturality of op^{Cell} , while the one on the right is obtained from the following square by whiskering with Cell on the left.

$$\begin{array}{ccc} \text{op Free Cell} & \xrightarrow{\text{op } \varepsilon} & \text{op} \\ \parallel & & \varepsilon \text{ op} \uparrow \\ \text{Free op Cell} & \xrightarrow{\text{Free op}^{\text{Cell}}} & \text{Free Cell op} \end{array}$$

Finally we can check that this square commutes by showing that two sides agree on every generator.

Definition 23. The opposite of an ω -category $(X, \alpha : TX \rightarrow X)$ with respect to some $w \in G$ is the ω -category consisting of the globular set $\text{op}_w X$ and the structure morphism

$$T \text{op}_w X \xrightarrow{\text{op}_{w,X}^T} \text{op}_w TX \xrightarrow{\text{op}_w \alpha} \text{op}_w X$$

The construction of the opposite of an ω -category is well-defined and gives rise to endofunctors

$$\text{op}_w : \omega \text{ Cat} \rightarrow \omega \text{ Cat}$$

for every $w \in G$ as shown by Street [24], and explained by Leinster [20, Theorem 6.1.1]. Moreover, the following lemma - an immediate consequence of Lemma 21 - shows that those endofunctors are invertible and give rise to an action

$$\text{op} : G \rightarrow \text{Aut}(\omega \text{ Cat})$$

of G on the category of ω -categories.

Lemma 24. *The natural transformation op_\emptyset^T is the identity, and for any $w, w' \in G$ the following diagram commutes:*

$$\begin{array}{ccc} T \text{op}_w \text{op}_{w'} & \xrightarrow{\text{op}_w^T \text{op}_{w'}^T} & \text{op}_w T \text{op}_{w'} \xrightarrow{\text{op}_w \text{op}_{w'}^T} \text{op}_w \text{op}_{w'} T \\ \parallel & & \parallel \\ T \text{op}_{ww'} & \xrightarrow{\text{op}_{ww'}^T} & \text{op}_{ww'} T \end{array}$$

The commutative diagram showing that op_w^T is a morphism of monads implies in particular that the components of the natural transformation $\text{op}_w^{\text{Cell}}$ are morphisms of free ω -categories, so it can also be seen as a natural isomorphism

$$\text{op}_w^K : \text{op}_w K^T \Rightarrow K^T \text{op}_w$$

The opposite functors on globular sets, computads and ω -categories are therefore related by the following five squares

$$\begin{array}{ccc} \omega \text{ Cat} \xrightarrow{\text{op}_w} \omega \text{ Cat} & \omega \text{ Cat} \xrightarrow{\text{op}_w} \omega \text{ Cat} & \text{Comp} \xrightarrow{\text{op}_w} \text{Comp} \\ U^T \downarrow & K^T \uparrow \text{op}_w^K \nearrow \uparrow K^T & \text{Free} \uparrow \\ \text{Glob} \xrightarrow{\text{op}_w} \text{Glob} & \text{Comp} \xrightarrow{\text{op}_w} \text{Comp} & \text{Glob} \xrightarrow{\text{op}_w} \text{Glob} \end{array}$$

$$\begin{array}{ccc}
\text{Comp} & \xrightarrow{\text{op}_w} & \text{Comp} \\
\text{Cell} \downarrow & \text{op}_w^{\text{Cell}} \nearrow & \downarrow \text{Cell} \\
\text{Glob} & \xrightarrow{\text{op}_w} & \text{Glob}
\end{array}
\qquad
\begin{array}{ccc}
\omega \text{ Cat} & \xrightarrow{\text{op}_w} & \omega \text{ Cat} \\
F^T \uparrow & \text{op}_w^F \nearrow & \uparrow F^T \\
\text{Glob} & \xrightarrow{\text{op}_w} & \text{Glob}
\end{array}$$

where

$$\text{op}_w^F = \text{op}_w^K \text{ Free} \qquad \text{op}_w^{\text{Cell}} = U^T \text{op}_w^K.$$

In particular, the opposite of an ω -category that is free on a globular set or computad is again free on the opposite of the underlying globular set or the opposite computad, up to natural isomorphism.

Example 25. In the light of the action of opposites on ω -categories, specialising Example 22 to the cell $\text{comp}_{1,0,2}$ can be interpreted as stating that the left whiskering of a cell y by a cell x in an ω -category \mathbb{X} can be described as the right whiskering of y by x in the opposite ω -category $\text{op}_1(\mathbb{X})$.

Remark 26. Since the free ω -category monad T is isomorphic to Leinster's initial contractible globular operad, it is equipped with a natural transformation $\alpha: T \Rightarrow T^{\text{str}}$ to the free strict ω -category monad admitting a contraction. An alternative way to construct the morphism of monads $\text{op}^T: T \text{ op} \Rightarrow \text{op} T$, would be to show that the composite

$$\text{op} T \text{ op} \xrightarrow{\text{op} \alpha \text{ op}} \text{op} T^{\text{str}} \text{ op} \xrightarrow{\text{op} \text{op}^{\text{str}}} T^{\text{str}}$$

is also a contractible operad. From this approach, it would have been harder to compute an explicit formula for op^T , which is needed for example to extend the proof assistant `catt` with a meta-operation computing the opposites, and it would not be immediately obvious that the property of being free on a computad is preserved by the formation of inverses.

5.4 Opposites of hom ω -categories

We will show that the operations of forming hom ω -categories and opposite categories commute. To make this statement precise, we first extend the action of $G = \mathbb{Z}_2^{\mathbb{N}_{>0}}$ on $\omega \text{ Cat}$ to an action on bipointed ω -categories

$$\text{op}: G \rightarrow \text{Aut}(\omega \text{ Cat}^{**})$$

by letting for $w \in G$ the opposite of a bipointed ω -category (X, x, x_+) be the ω -category $\text{op}_w X$ with the same basepoints when $1 \notin w$, and with the basepoints swapped when $1 \in w$.

Lemma 27. *For every $w \in G$, there exists a natural isomorphism*

$$\text{op}_w^\Sigma: \Sigma \text{op}_{w-1} \Rightarrow \text{op}_w \Sigma: \text{Comp} \rightarrow \text{Comp}^{**}$$

compatible with the natural isomorphism of Lemma 13 in the sense that

$$\text{op}_w^\Sigma \text{ Free} = \text{Free} \text{op}_w^\Sigma$$

and the following diagram commutes:

$$\begin{array}{ccc}
\Sigma \text{op}_{w-1} \text{Cell} & \xrightarrow{\text{op}_w^\Sigma \text{Cell}} & \text{op}_w \Sigma \text{Cell} \\
\Sigma \text{op}_{w-1}^{\text{Cell}} \Downarrow & & \Downarrow \text{op}_w \Sigma^{\text{Cell}} \\
\Sigma \text{Cell op}_{w-1} & & \text{op}_w \text{Cell } \Sigma \\
\Sigma^{\text{Cell}} \text{op}_{w-1} \Downarrow & & \Downarrow \text{op}_w^{\text{Cell}} \Sigma \\
\text{Cell } \Sigma \text{op}_{w-1} & \xrightarrow[\text{Cell op}_w^\Sigma]{} & \text{Cell op}_w \Sigma
\end{array}$$

Proof. We will build natural isomorphism

$$\text{op}_w^\Sigma : \Sigma \text{op}_{w-1} \Rightarrow \text{op}_w \Sigma : \text{Comp}_n \rightarrow \text{Comp}_{n+1}$$

inductively on $n \geq -1$ commuting with the forgetful functors u_n , the inclusion functors Free_n and making the following pentagons commute

$$\begin{array}{ccc}
& \text{Cell}_n & \\
\text{op}_{w-1}^{\text{Cell}} \swarrow & & \searrow \Sigma^{\text{Cell}} \\
\text{Cell}_n \text{op}_{w-1} & & \text{Cell}_{n+1} \Sigma \\
\Sigma^{\text{Cell}} \text{op}_{w-1} \Downarrow & & \Downarrow \text{op}_w^{\text{Cell}} \Sigma \\
\text{Cell}_{n+1} \Sigma \text{op}_{w-1} & \xrightarrow{\text{Cell}_{n+1} \text{op}_w^\Sigma} & \text{Cell}_{n+1} \text{op}_w \Sigma
\end{array}$$

$$\begin{array}{ccc}
& \text{Sph}_n & \\
\text{op}_{w-1}^{\text{Sph}} \swarrow & & \searrow \Sigma^{\text{Sph}} \\
\text{Sph}_n \text{op}_{w-1} & & \text{Sph}_{n+1} \Sigma \\
\Sigma^{\text{Sph}} \text{op}_{w-1} \Downarrow & & \Downarrow \text{op}_w^{\text{Sph}} \Sigma \\
\text{Sph}_{n+1} \Sigma \text{op}_{w-1} & \xrightarrow{\text{Sph}_{n+1} \text{op}_w^\Sigma} & \text{Sph}_{n+1} \text{op}_w \Sigma
\end{array}$$

For the unique (-1) -computad, we let

$$\text{op}_w^\Sigma : \{v_-, v_+\} \rightarrow \{v_-, v_+\}$$

be the identity function when $1 \notin w$, and the function swapping the two generators when $1 \in w$. This is a natural isomorphism making the second pentagon commute: both sides send the unique (-1) -sphere to the 0-sphere (v_-, v_+) when $1 \notin w$, and to the 0-sphere (v_+, v_-) otherwise.

For an n -computad $C = (C_{n-1}, V_n^C, \phi_n^C)$ where $n \in \mathbb{N}$, we let

$$\text{op}_{w,C}^\Sigma = (\text{op}_{w,C_{n-1}}^\Sigma, \text{var}) : \Sigma \text{op}_{w-1} C \rightarrow \text{op}_w \Sigma C$$

This is a well-defined morphism of computads by the commutativity of the second pentagon one dimension lower. Moreover, it commutes with the forgetful and the free functors by construction.

We will show that the first pentagon commutes for a computad C and that op_w^Σ is natural mutually inductively. First we see that the pentagon commutes when restricted to generators, since both $\text{op}_w^{\text{Cell}}$ and Σ^{Cell} preserve generators. Suppose now that $n > 0$ and let $c = \text{coh}(B, A, \tau)$ a coherence n -cell of C . Then we see that

$$\begin{aligned} & (\text{op}_{w, \Sigma C}^{\text{Cell}} \circ \Sigma_C^{\text{Cell}})(c) \\ &= \text{coh}(\text{op}_w \Sigma B, A_1, \text{op}_w \Sigma \tau \circ \text{Free}_n \text{op}_w^\Sigma B) \\ (\text{Cell}_{n+1}(\text{op}_{w, C}^\Sigma) \circ \Sigma_{\text{op}_{w-1} C}^{\text{Sph}} \circ \text{op}_{w-1, C}^{\text{Sph}})(c) \\ &= \text{coh}(\Sigma \text{op}_{w-1} B, A_2, \text{op}_{w, C}^\Sigma \circ \Sigma \text{op}_{w-1} \tau \circ \Sigma \text{Free}_n \text{op}_{w-1}^B) \end{aligned}$$

where

$$\begin{aligned} A_1 &= \text{Sph}_n \text{Free}_n (\text{op}_w^\Sigma B)^{-1} (\text{op}_w^{\text{Sph}} \Sigma^{\text{Sph}} A) \\ A_2 &= \Sigma_{\text{op}_{w-1} C}^{\text{Sph}} (\text{Sph}_{n-1} \text{Free}_{n-1} (\text{op}_{w-1}^B)^{-1} (\text{op}_{w-1}^{\text{Sph}} A)) \end{aligned}$$

By definition of the suspension and the opposite of a tree, we have that

$$\text{op}_w \Sigma B = \Sigma \text{op}_{w-1} B,$$

so the trees over which those coherence cells are built agree. Moreover, the natural isomorphism $\text{op}_w^\Sigma B$ is defined to be the composite

$$\text{op}_w^\Sigma B = \text{op}_{w, \text{Pos}(B)}^\Sigma \circ \Sigma \text{op}_{w-1}^B.$$

Using that fact and the naturality of Σ^{Sph} , we can rewrite the spheres A_1 and A_2 respectively as

$$\begin{aligned} A_1 &= \text{Sph}_n \text{Free}_n \Sigma (\text{op}_{w-1}^B)^{-1} (\text{Sph}_n \text{Free}_n (\text{op}_{w, \text{Pos}(B)}^\Sigma)^{-1} (\text{op}_w^{\text{Sph}} \Sigma^{\text{Sph}} A)) \\ A_2 &= \text{Sph}_n \Sigma \text{Free}_{n-1} (\text{op}_{w-1}^B)^{-1} (\Sigma_{\text{op}_{w-1} C}^{\text{Sph}} \text{op}_{w-1}^{\text{Sph}} A) \end{aligned}$$

and observe that they agree by commutativity of the diagram for spheres one dimension lower, and commutativity of the suspension with the functor Free_n . Moreover, we may assume that the naturality square for τ commutes by the inductive hypothesis, which shows that the morphisms defining the coherence cells agree. Therefore, the first pentagon commutes on coherence cells as well.

Let now $\sigma : D \rightarrow C$ be a morphism of n -computads and suppose that the pentagon commutes when restricted to cells of the form $\sigma_{n, V}(v)$ for $v \in V_n^D$. To show that the naturality square for σ commutes, i.e. that

$$\text{op}_{w, C}^\Sigma \circ \Sigma \text{op}_{w-1} \sigma = \text{op}_w \Sigma \sigma \circ \text{op}_{w, D}^\Sigma,$$

we may assume by induction on the dimension and commutativity with the forgetful functors that the underlying morphisms of n -computads agree. It

remains to show that the two morphisms agree on top-dimensional generators. Let therefore $v \in V_{n+1}^{\Sigma \text{op}_{w-1} D} = V_n^D$ be a generator. Then

$$\begin{aligned}
(\text{op}_{w,C}^{\Sigma} \circ \Sigma \text{op}_{w-1} \sigma)_V(v) &= \text{Cell}_{n+1}(\text{op}_{w,C}^{\Sigma})(\Sigma_{\text{op}_{w-1,C}}^{\text{Cell}} \text{op}_{w-1,C}^{\text{Cell}}(\sigma_V(v))) \\
&= \text{op}_{w,\Sigma C}^{\text{Cell}} \Sigma_C^{\text{Cell}}(\sigma_V(v)) \\
&= (\text{op}_w \Sigma \sigma)_V(v) \\
&= \text{Cell}_{n+1}(\text{op}_w \Sigma \sigma)(\text{var } v) \\
&= \text{Cell}_{n+1}(\text{op}_w \Sigma \sigma)((\text{op}_{w,D}^{\Sigma})_V v) \\
&= (\text{op}_w \Sigma \sigma \circ \text{op}_{w,D}^{\Sigma})_V(v)
\end{aligned}$$

so the two morphisms agree on generators as well. Hence, the naturality square commutes.

This concludes the induction on $n \in \mathbb{N}$. By commutativity with the forgetful functors, the natural isomorphisms op_w^{Σ} for every $n \in \mathbb{N}$ combine to a natural isomorphism

$$\text{op}_w^{\Sigma} : \Sigma \text{op}_{w-1} \Rightarrow \text{op}_w \Sigma : \text{Comp} \rightarrow \text{Comp}$$

as well. Commutativity of the first pentagon shows that the diagram of the lemma commutes, since $\text{op}_w^{\Sigma} \text{Cell}$ is the identity on positive-dimensional cells. \square

Proposition 28. *For every $w \in G$, the following diagram commutes*

$$\begin{array}{ccc}
\omega \text{Cat}^{**} & \xrightarrow{\text{op}_w} & \omega \text{Cat}^{**} \\
\Omega \downarrow & & \downarrow \Omega \\
\omega \text{Cat} & \xrightarrow{\text{op}_{w-1}} & \omega \text{Cat}
\end{array}$$

Proof. In order to prove commutativity of this diagram for some $w \in G$, it is useful to prove commutativity of the analogous diagram on the level of globular sets first:

$$\begin{array}{ccc}
\text{Glob}^{**} & \xrightarrow{\text{op}_w} & \text{Glob}^{**} \\
\Omega \downarrow & & \downarrow \Omega \\
\text{Glob} & \xrightarrow{\text{op}_{w-1}} & \text{Glob}
\end{array}$$

The mate of the natural isomorphism of Lemma 13 is a natural transformation fitting in this square defined as the whiskered composite

$$\text{op}_{w-1} \Omega = \Omega \Sigma \text{op}_{w-1} \Omega \xrightarrow{\Omega \text{op}_w^{\Sigma} \Omega} \Omega \text{op}_w \Sigma \Omega \xrightarrow{\Omega \text{op}_w \kappa} \Omega \text{op}_w$$

for κ the counit of the adjunction $\Sigma \dashv \Omega$. One of the snake equations of this adjunction states that $\Omega \kappa$ is an identity. Combining that with the fact that op_w preserves cells and acts trivially on morphisms, we see that $\Omega \text{op}_w \kappa$ must also be an identity. Moreover, the natural isomorphism op_w^{Σ} was defined to be the

identity on positive-dimensional cells, so $\Omega \text{op}_w^\Sigma$ must also be an identity. Since the mate of op_w^Σ is an identity natural transformation, we conclude that the square above must commute.

Since the diagram commutes on the level of globular sets, and the forgetful functors U^T and $U^{T^{**}}$ are faithful, it follows that the diagram commutes on the level of ω -categories if it commutes for objects, meaning that for every bipointed ω -category X , the ω -categories $\Omega \text{op}_w X$ and $\text{op}_{w-1} \Omega X$ are equal. Both ω -categories have the same underlying globular set by commutativity of the square on the level of globular sets, so it remains to show that they have the same structure morphisms. Unwrapping the definitions of op_w and Ω , this amounts to the commutativity of the following diagram of natural transformations

$$\begin{array}{ccc} T \text{op}_{w-1} \Omega & \xrightarrow{\text{op}_{w-1}^T \Omega} & \text{op}_{w-1} T \Omega \xrightarrow{\Omega^T} \text{op}_{w-1} \Omega T^{**} \\ \parallel & & \parallel \\ T \Omega \text{op}_w & \xrightarrow{\Omega^T \text{op}_w} & \Omega T^{**} \text{op}_w \xrightarrow{\Omega \text{op}_w^{T^{**}}} \Omega \text{op}_w T^{**} \end{array}$$

where $\text{op}_w^{T^{**}}$ is simply op_w^T seen as a natural isomorphism between bipointed globular sets. By naturality of the mate correspondence, commutativity of this diagram is equivalent to that of the following one:

$$\begin{array}{ccc} \Sigma T \text{op}_{w-1} & \xrightarrow{\Sigma \text{op}_{w-1}^T} & \Sigma \text{op}_{w-1} T \xrightarrow{\text{op}_w^\Sigma T} \text{op}_w \Sigma T \\ \Sigma^T \text{op}_{w-1} \Downarrow & & \Downarrow \text{op}_w \Sigma^T \\ T^{**} \Sigma \text{op}_{w-1} & \xrightarrow{T^{**} \text{op}_w^\Sigma} & T^{**} \text{op}_w \Sigma \xrightarrow{\text{op}_w^{T^{**}} \Sigma} \text{op}_w T^{**} \Sigma \end{array}$$

Replacing each op_w^T by each inverse and rotating the diagram, we are left to show that the following diagram commutes:

$$\begin{array}{ccc} \Sigma \text{op}_{w-1} T & \xrightarrow{\Sigma \text{op}_{w-1}^{\text{Cell Free}}} & \Sigma T \text{op}_{w-1} \xrightarrow{\Sigma^{\text{Cell Free}} \text{op}_{w-1}} T^{**} \Sigma \text{op}_{w-1} \\ \text{op}_w^\Sigma T \Downarrow & & \Downarrow T^{**} \text{op}_w^\Sigma \\ \text{op}_w \Sigma T & \xrightarrow{\text{op}_w \Sigma^{\text{Cell Free}}} & \text{op}_w T^{**} \Sigma \xrightarrow{\text{op}_w^{\text{Cell}^{**}} \text{Free}^{**} \Sigma} T^{**} \text{op}_w \Sigma \end{array}$$

But this is precisely the diagram of Lemma 27 whiskered on the right with Free , hence it commutes. \square

6 Applications to Eckmann-Hilton cells

Throughout this article, we have illustrated the construction of suspensions and opposites on the various binary composition operations that have introduced in Section 3.1. We will illustrate further our constructions in action on a more complex construction, that of the Eckmann-Hilton cells.

6.1 A clockwise Eckmann-Hilton cell

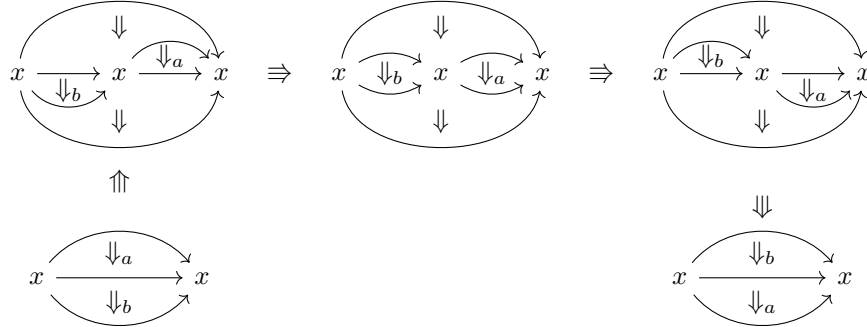
The Eckmann-Hilton argument states that given two monoid structures on a set satisfying some compatibility relation, the two structures are necessarily equal and commutative. This argument can be translated to weak ω -categories, where it states that considering two cells c and c' whose boundaries are both identities over the same cell, there exists a higher cell witnessing the commutativity of their vertical composite. This cell comes from the compatibility of the compositions $*_1$ and $*_0$ on 1-cells of the ω -category.

We sketch here the definition of such a higher cell, that we call the clockwise Eckmann-Hilton argument. For this, we consider the computad C_{eh} which has one 0-dimensional generator x and two 2-dimensional generators a, b whose sources and targets are all the identity cell id_x on x . We will define a 3-cell eh in C_{eh} whose source is $a *_1 b$ and whose target is $b *_1 a$. Given a weak ω -category \mathbb{X} , a pair of 2-cells c, c' in \mathbb{X} whose sources and targets are identities on the same 0-cell correspond exactly to a morphism of ω -categories $f : K^T C_{\text{eh}} \rightarrow \mathbb{X}$. The clockwise Eckmann-Hilton cell on c and c' in \mathbb{X} is then defined to be:

$$\text{eh}(c, c') := f(\text{eh}).$$

The source of this cell is $c *_1 c'$ and its target is $c' *_1 c$.

The 3-cell eh can be geometrically illustrated as the following composite



where the unlabelled 1-cells are the identity on x and the unlabelled 2-cells are unitors. The 3-cells appearing in the diagrams are composites of unitors, associators and interchangers. The precise definition of this cell is quite cumbersome and beyond the scope of our paper. It has been formalised in the proof assistant `catt` [12] for working in finite computads [8]. We note that the Eckmann-Hilton cell is not unique, since we can get cells of the same type by iteratively rotating the cells a and b around each other.

6.2 Opposites of the Eckmann-Hilton cell

We now discuss the action of the opposite operation on the clockwise Eckmann-Hilton cell. The first thing that one can notice is that the computad C_{eh} is

self-dual, in that $\text{op}_w C_{\text{eh}} = C_{\text{eh}}$ for every $w \in G$. Moreover, we have the following equations for the composites of a and b :

$$\begin{aligned} \text{op}_1^{\text{Cell}}(a *_0 b) &= b *_0 a & \text{op}_1^{\text{Cell}}(a *_1 b) &= a *_1 b \\ \text{op}_2^{\text{Cell}}(a *_0 b) &= a *_0 b & \text{op}_2^{\text{Cell}}(a *_1 b) &= b *_1 a. \end{aligned}$$

It follows that the opposites $\text{op}_1^{\text{Cell}}(\text{eh})$ and $\text{op}_2^{\text{Cell}}(\text{eh})$ are again Eckmann-Hilton cells, obtained by rotating a and b around each other anticlockwise. They further satisfy the following equality

$$\text{op}_1^{\text{Cell}}(\text{eh}(a, b)) = \text{op}_2^{\text{Cell}}(\text{eh}(b, a)),$$

which we have formally verified with the proof assistant `catt`. Given an ω -category \mathbb{X} and two cells c, c' of \mathbb{X} whose sources and targets are identities on the same 0-cell, the anticlockwise Eckmann-Hilton cell on c and c' in \mathbb{X} can be obtained either as the clockwise Eckmann-Hilton cell on c and c' in the opposite category $\text{op}_1 \mathbb{X}$, or as the clockwise Eckmann-Hilton cell on c' and c in the opposite category $\text{op}_2 \mathbb{X}$.

Remark 29. It turns out that the opposite of the Eckmann-Hilton cell $\text{op}_1(\text{eh})$ is also its inverse, as explained in our subsequent work [7].

6.3 Suspensions of the Eckmann-Hilton cell

Finally, we discuss the construction of the suspension applied to the Eckmann-Hilton cell. The computad $\Sigma^n C_{\text{eh}}$ can be described explicitly, and it is immediate to see that a morphism of ω -categories $K^T \Sigma^n C_{\text{eh}} \rightarrow \mathbb{X}$ exactly corresponds to a pair of $(n+2)$ -cells c and c' in \mathbb{X} , whose sources and targets are all identities over the same n -cell. In this situation, we define the clockwise Eckmann-Hilton cell on c and c' to be

$$\text{eh}_n(c, c') := f((\Sigma^{\text{Cell}})^n(\text{eh})).$$

This cell witnesses the equivalence between $c *_n c'$ and $c' *_n c$, and comes from the interchanger between the $(n+1)$ -composition and the n -composition of $(n+2)$ -cells. This witness cell is obtained as the clockwise Eckmann-Hilton cell on c and c' in the iterated hom ω -category on \mathbb{X} .

Acknowledgements

We would like to thank Prof. Jamie Vicary for his support during this project. We would also like to thank Prof. John Bourke and Dimitri Ara for their helpful remarks on earlier versions of this work. Furthermore, the second author would like to acknowledge funding from the Onassis foundation - Scholarship ID: F ZQ 039-1/2020-2021.

References

- [1] Jiří Adámek and Jiří Rosický. *Locally Presentable and Accessible Categories*. Cambridge University Press, 1 edition, March 1994. doi:10.1017/CB09780511600579.
- [2] Thorsten Altenkirch and Ondrej Rypacek. A syntactical approach to weak omega-groupoids. In *Computer Science Logic (CSL'12) - 26th International Workshop/21st Annual Conference of the EACSL (CSL 2012)*, page 15 pages. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik GmbH, Wadern/Saarbruecken, Germany, 2012. doi:10.4230/LIPICS.CSL.2012.16.
- [3] Bruce Bartlett, Christopher L. Douglas, Christopher J. Schommer-Pries, and Jamie Vicary. Modular categories as representations of the 3-dimensional bordism 2-category, September 2015. arXiv:1509.06811.
- [4] Michael A. Batanin. Computads for finitary monads on globular sets. In Ezra Getzler and Mikhail Kapranov, editors, *Higher Category Theory*, volume 230 of *Contemporary Mathematics*, pages 37–57. American Mathematical Society, Providence, Rhode Island, 1998. doi:10.1090/conm/230/03337.
- [5] Michael A. Batanin. Monoidal globular categories as a natural environment for the theory of weak n -categories. *Advances in Mathematics*, 136(1):39–103, June 1998. doi:10.1006/aima.1998.1724.
- [6] Thibaut Benjamin. *A type theoretic approach to weak ω -categories and related higher structures*. Thèse de doctorat, Institut Polytechnique de Paris, November 2020.
- [7] Thibaut Benjamin and Ioannis Markakis. Invertible cells in ω -categories, June 2024. arXiv:2406.12127.
- [8] Thibaut Benjamin, Ioannis Markakis, and Chiara Sarti. Catt contexts are finite computads. In *Proceedings of the 40th Conference on Mathematical Foundations of Programming Semantics*, Oxford, UK, 2024. arXiv:2405.00398.
- [9] Clemens Berger. A cellular nerve for higher categories. *Advances in Mathematics*, 169(1):118–175, July 2002. doi:10.1006/aima.2001.2056.
- [10] Thomas Cottrell and Soichiro Fujii. Hom weak ω -categories of a weak ω -category. *Mathematical Structures in Computer Science*, 32(4):420–441, April 2022. doi:10.1017/S0960129522000111.
- [11] Christopher J. Dean, Eric Finster, Ioannis Markakis, David Reutter, and Jamie Vicary. Computads for weak ω -categories as an inductive type. *Advances in Mathematics*, 450:109739, July 2024. doi:10.1016/j.aim.2024.109739.

- [12] Eric Finster and Samuel Mimram. A type-theoretical definition of weak ω -categories. In *Proceedings of the 32nd Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 1–12, Reyjavik Iceland, June 2017. ACM. [arXiv:1706.02866](#), [doi:10.5555/3329995.3330059](#).
- [13] Richard Garner. Homomorphisms of higher categories. *Advances in Mathematics*, 224(6):2269–2311, August 2010. [arXiv:0810.4450](#), [doi:10.1016/j.aim.2010.01.022](#).
- [14] Alexander Grothendieck. Pursuing stacks, 1983. [arXiv:2111.01000](#).
- [15] Simon Henry and Edoardo Lanari. On the homotopy hypothesis in dimension 3. *Theory and Applications of Categories*, 39:735–768, 2023.
- [16] P. T. Johnstone. Adjoint lifting theorems for categories of algebras. *Bulletin of the London Mathematical Society*, 7(3):294–297, November 1975. [doi:10.1112/blms/7.3.294](#).
- [17] Gregory M. Kelly and Ross Street. Review of the elements of 2-categories. In *Category Seminar*, volume 420, pages 75–103, Berlin, Heidelberg, 1974. Springer Berlin Heidelberg. [doi:10.1007/BFb0063101](#).
- [18] Yves Lafont and François Métayer. Polygraphic resolutions and homology of monoids. *Journal of Pure and Applied Algebra*, 213(6):947–968, June 2009. [doi:10.1016/j.jpaa.2008.10.005](#).
- [19] Yves Lafont, François Métayer, and Krzysztof Worytkiewicz. A folk model structure on omega-cat. *Advances in Mathematics*, 224(3):1183–1231, June 2010. [arXiv:0712.0617](#), [doi:10.1016/j.aim.2010.01.007](#).
- [20] Tom Leinster. *Higher operads, higher categories*. Number 298 in London Mathematical Society lecture note series. Cambridge University Press, Cambridge, UK ; New York, 2004. [arXiv:math/0305049](#).
- [21] Peter LeFanu Lumsdaine. Weak ω -categories from intensional type theory. In Pierre-Louis Curien, editor, *Typed Lambda Calculi and Applications*, volume 5608, pages 172–187. Springer Berlin Heidelberg, Berlin, Heidelberg, 2009. [doi:10.1007/978-3-642-02273-9_14](#).
- [22] Samuel Mimram. Towards 3-dimensional rewriting theory. *Logical Methods in Computer Science*, 10(2):1, April 2014. [arXiv:1403.4094](#), [doi:10.2168/LMCS-10\(2:1\)2014](#).
- [23] François Métayer. Cofibrant objects among higher-dimensional categories. *Homology, Homotopy and Applications*, 10(1):181–203, 2008. [doi:10.4310/HHA.2008.v10.n1.a7](#).
- [24] Ross Street. The formal theory of monads. *Journal of Pure and Applied Algebra*, 2(2):149–168, July 1972. [doi:10.1016/0022-4049\(72\)90019-9](#).

- [25] Benno van den Berg and Richard Garner. Types are weak ω -groupoids. *Proceedings of the London Mathematical Society*, 102(2):370–394, February 2011. [arXiv:0812.0298](#), [doi:10.1112/plms/pdq026](#).