

Higher Eckmann-Hilton Arguments in Type Theory

Thibaut Benjamin* Ioannis Markakis† Wilfred Offord‡
 Chiara Sarti§ Jamie Vicary¶

Department of Computer Science and Technology,
 University of Cambridge, Cambridge, UK

29th January 2025

Abstract

We use a type theory for ω -categories to produce higher-dimensional generalisations of the Eckmann-Hilton argument. The heart of our construction is a family of padding and repadding techniques, which give a notion of congruence between cells of different types. This gives explicit witnesses in all dimensions that, for cells with degenerate boundary, all composition operations are congruent and commutative. Our work has been implemented, allowing us to explicitly compute these witnesses, and we show these grow rapidly in complexity as the parameters are varied. Our results can also be exported as elements of identity types in Martin-Löf type theory, and hence are of relevance in homotopy type theory.

Keywords: dependent type theory, Eckmann-Hilton, homotopy type theory, higher categories, formalisation

1 Introduction

1.1 Overview

Higher category theory is a flexible language, which increasingly finds application across a range of areas, including topological field theory [29], term rewriting [4] and quantum computing [1]. Notably, it is an area of mathematics where type-theoretic ideas have had considerable impact [24, 28]. We work

*thibaut.benjamin@polytechnique.edu

†ioannis.markakis@cl.cam.ac.uk

‡wgo2@cam.ac.uk

§cs2197@cam.ac.uk

¶jamie.vicary@cl.cam.ac.uk

in the *globular* setting, where for $n > 0$ an n -cell a has source and target $(n-1)$ -cells $\partial^-(a), \partial^+(a)$, which we often express with the type-theoretic notation $a : \partial^-(a) \rightarrow \partial^+(a)$. For $n > 1$ we impose the *globularity condition*, that whenever $a : u \rightarrow v$, then u and v have the same type. We may further define the k -dimensional source and target $\partial_k^-(a), \partial_k^+(a)$ for any $0 \leq k < n$ by taking successive boundaries.

The structure of a higher category allows n -cells to be composed in a variety of ways. In particular, if a, b are n -cells with $\partial_k^+(a) = \partial_k^-(b)$, for $0 \leq k < n$ we may form their *binary k -composite* $a *_k b$, which we understand as “gluing” a and b along their common k -dimensional boundary. We illustrate this in Figures 1a and 1b, which show the 1- and 0-composites, respectively, of a pair of 2-cells.

The axioms of a higher category also give us access to a class of cells, called *coherences*, that serve as witnesses that their source and target are in some sense “equivalent”. To give some first examples, given a 0-cell x , we may construct the *identity*, a 1-cell $\text{id}_x : x \rightarrow x$, and the *unbiased unitor*, a 2-cell $u_x : \text{id}_x *_0 \text{id}_x \rightarrow \text{id}_x$; and given a 1-cell $f : x \rightarrow y$, we may construct the *left unitor* $\lambda_f : \text{id}_x *_0 f \rightarrow f$, and the *right unitor* $\rho_f : f *_0 \text{id}_x \rightarrow f$. All such coherences are part of a broader class of cells called *equivalences*, which are cells with *inverses* satisfying a weak cancellation law, up to higher equivalences. For an equivalence $e : u \rightarrow v$, we denote its inverse $e^{-1} : v \rightarrow u$.

In a higher category, there is no reason to expect that the composition operations are related. For example, when they are both defined, we do not in general expect an equivalence between $a *_0 b$ and $a *_1 b$; indeed testing these terms for equivalence would not usually make sense, as they have different types. However, if the cell boundaries are *degenerate*—that is, given by identities—we can make this question well-posed by composing $a *_0 b$ with the coherences u_x and u_x^{-1} , to change its boundary, a procedure which we call *padding*. Perhaps surprisingly, we can then construct an equivalence $H(a, b) : a *_1 b \rightarrow u_x^{-1} *_1 (a *_0 b) *_1 u_x$, which we illustrate in Figure 2.

More generally, to extend the notion of equivalence to cells with different types, we define *congruence* as the smallest equivalence relation on n -cells with the following properties: an n -cell is congruent to its composite with any coherence, and equivalent cells are congruent. The argument above then shows that $a *_1 b$ and $a *_0 b$ are congruent.

Furthermore, applying a duality construction, we obtain another equivalence $H^{\text{op}}(a, b) : a *_1 b \rightarrow u_x^{-1} *_1 (b *_0 a) *_1 u_x$, and we may compose to obtain the



Figure 1: Composites of a pair of 2-cells.

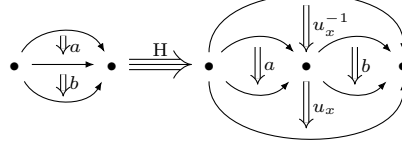


Figure 2: The equivalence $H : a *_1 b \rightarrow u_x^{-1} *_1 (a *_0 b) *_1 u_x$.

following equivalence:

$$EH(a, b) := H(a, b) \star_2 (H^{\text{op}}(b, a))^{-1} : a *_1 b \rightarrow b *_1 a$$

Hence we conclude that the operation $*_1$ is commutative up to equivalence. This construction is well-known, and can be seen as a categorification of the classical Eckmann-Hilton argument, which has important homotopy-theoretic consequences, for instance that higher homotopy groups are abelian [19, 23].

In this paper, we use a type-theoretic definition of ω -category to generalise this construction, producing a parameterised family of equivalences as follows, where a, b are n -cells with degenerate boundaries,¹ and where $0 \leq k, l < n$ are distinct composition directions:

$$H_{k,l}^n(a, b) : a *_k b \rightarrow \Theta_{k,l}^n(a *_l b)$$

Here $\Theta_{p,q}^n$ is a *padding* operation, which composes its argument with coherences, generalising our earlier use of u_x and u_x^{-1} . It follows that $a *_k b$ and $a *_l b$ are congruent.

As a corollary, applying a similar duality operation, we can produce another $(n+1)$ -cell with type $b *_k a \rightarrow \Theta_{k,l}^n(a *_l b)$, and compose as above to obtain an equivalence as follows:

$$EH_{k,l}^n(a, b) : a *_k b \rightarrow b *_k a$$

This witnesses that composition of n -cells with degenerate boundary is commutative in any choice of direction k , up to equivalence. Such a construction can reasonably be described as a *higher Eckmann-Hilton argument*. Furthermore, we have a family of such witnesses, according to the choice of the second parameter l .

Furthermore, our $H_{k,l}^n$ construction can be extended to the case where paddings can appear on both sides, yielding cells $H_{p,k,l}^n$ as follows:

$$H_{p,k,l}^n : \Theta_{p,k}^n(a *_k b) \rightarrow \Theta_{p,l}^n(a *_l b)$$

¹For $n > 1$, an n -cell has *degenerate boundary* when its source and target both take the form $\text{id}_x^{n-1} := \text{id}(\dots(\text{id}_x)\dots)$, where x is a 0-cell.

When $p = k$ the operation $\Theta_{p,k}^n(-)$ becomes the identity, and so this construction is an extension of our main result, with $H_{k,k,l}^n = H_{k,l}^n$ and $H_{l,k,l}^n = (H_{k,l}^n)^{-1}$. These cells $H_{p,k,l}^n$ together with their opposites can be arranged in a structure we call the *Eckmann-Hilton n -sphere*, which we illustrate for $n = 3$ in Fig. 3. This structure has been studied before in semi-strict models [16], but not in the fully weak setting.

We work in the type theory CaTT due to Finster and Mimram [20], whose models correspond to Batanin-Leinster ω -categories [3, 11, 14]. This theory has an interpretation in homotopy type theory [7], and as a result our constructions can be exported as inhabitants of identity types. Our work therefore has relevance to synthetic homotopy theory. The Eckmann-Hilton cell $\text{EH}_{1,0}^2$ has been previously constructed in homotopy type theory, which easily yields $\text{EH}_{n-1,n-2}^n$ by changing the base type. For the remaining cases of $\text{EH}_{k,l}^n$, we believe this is the first explicit algebraic construction.

Furthermore, our setting is more general than that of homotopy type theory in two ways. Firstly, it is fully weak, containing nontrivial equivalences such as $\text{id}_x \rightarrow \text{id}_x *_0 \text{id}_x$, which are computational equalities in homotopy type theory. Secondly, it is directed, and thus applies in the case where the cells a and b are not themselves invertible.

The type theory CaTT has been implemented as a proof assistant, in which we have automated our construction as a meta-operation. This allows us to evaluate the terms $H_{k,l}^n$ and quantify their complexity. In some directions of the parameter space, for instance as $n - \max\{k, l\}$ increases, the complexity of generating and type-checking these terms grows rapidly, exhausting available memory on a typical workstation after the first several cases. We illustrate the term sizes for the tractable cases in Figure 4. Due to their intricate structure, defining these terms by hand would likely not be feasible.

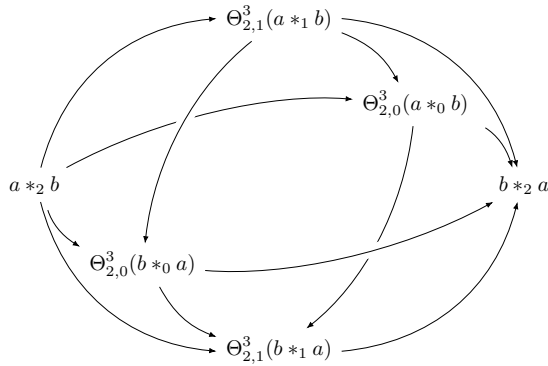


Figure 3: The 3-dimensional Eckmann-Hilton sphere.

1.2 Related Work

The type theory CaTT was introduced by Finster and Mimram [20], and subsequently studied by Benjamin [6] and Benjamin, Mimram and Finster [11], who in particular established an equivalence between its type-theoretic models and ω -categories as defined by Batanin and Leinster [5, 25]. The type-theoretical approach to higher category theory is closely related to the *computadic* approach [8, 9, 17], and indeed Benjamin, Markakis and Sarti [10] showed that contexts of CaTT are equivalent to finite computads, i.e. presentations of free ω -categories on a finite signature. Thanks to this, many results for computads carry over directly to our setting, and we will freely quote their implications for CaTT . In particular, this previous work allows the constructions of various meta-operations on the terms of CaTT , which we will rely on for our construction.

Our *padding* technique, which we use to construct the operations $\Theta_{k,l}^n$, is inspired by similar constructions in the work of Finster et al. [21] and Fujii, Hoshino and Maehara [22], and includes these as a special case.

1.3 Technical Approach

Our main result is Theorem 4.4, which demonstrates congruence for $a *_k b$ and $a *_l b$ when a, b are degenerate, by constructing an equivalence as follows:

$$H_{k,l}^n : a *_k b \rightarrow \Theta_{k,l}^n(a *_l b)$$

These cells are constructed by induction, and the primary technical obstacle is definition of the base cases $H_{n-1,0}^n$ and $H_{0,n-1}^n$. Once these cases are established, we obtain the remaining cases using a *naturality* operation which allows us to construct $H_{k,l}^{n+1}$ from $H_{k,l}^n$, and a *suspension* operation which allows us to construct $H_{k+1,l+1}^{n+1}$ from $H_{k,l}^n$.

Here we give an outline of our approach to defining the base cases, to serve as an informal preview of the full construction given in Section 3. The base case $H_{n-1,0}^n$ is constructed as the following composite, which we illustrate in Fig. 5:

$$H_{n-1,0}^n = X_1 *_n X_2 *_n X_3 *_n X_4$$

We now examine these steps in turn.

Step 1. Application of Generalised Unitors. In the first step, our goal is to apply equivalences to transform the cells a and b into their identity composites

n	$H_{1,0}^n$	$H_{2,1}^n$	$H_{3,2}^n$	$H_{2,0}^n$	$H_{3,1}^n$	$H_{3,0}^n$
2	5,340					
3	67,208	6,993		44,209		
4	5,339,606	116,343	8,152	3,117,243	73,981	2,615,998
5	(overflow)	(overflow)	178,592	(overflow)	6,176,548	(overflow)

Figure 4: Character counts for selected artifacts $H_{k,l}^n$.

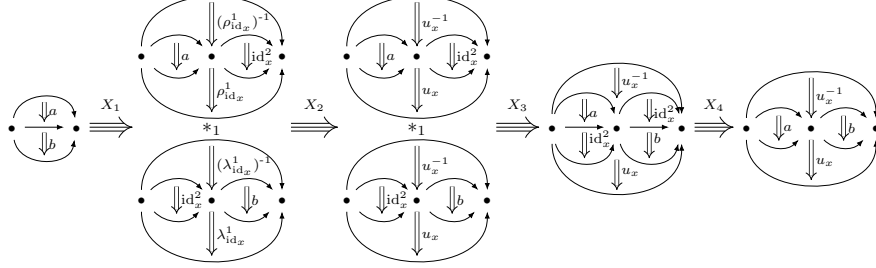


Figure 5: Construction of the base cases $H_{n-1,0}^n$ for the case $n = 2$.

$a *_0 \text{id}_x^n$ and $\text{id}_x^n *_0 b$, respectively. This cannot be done directly, as these cells do not have the same boundary. Instead we build coherences as follows, which we call *generalised unitors*:

$$\begin{aligned} \rho_a^n &: \Theta_\rho^n(a *_0 \text{id}_x^n) \rightarrow a \\ \lambda_b^n &: \Theta_\lambda^n(\text{id}_x^n *_0 b) \rightarrow b \end{aligned}$$

Here $\Theta_\rho^n(-)$ and $\Theta_\lambda^n(-)$ are operations which iteratively compose their argument with generalised right or left unitors respectively. This yields the following:

$$X_1 = (\rho_a^n)^{-1} *_{n-1} (\lambda_b^n)^{-1} : a *_{n-1} b \rightarrow \Theta_\rho^n(a *_0 \text{id}_x^n) *_{n-1} \Theta_\lambda^n(\text{id}_x^n *_0 b)$$

Step 2. Repadding. For the next step, we apply equivalences which modify the generalised left and right unitors, transforming them into generalised *unbiased* unitors. This “unbiasing” process is related to the familiar coherence equation $\rho_I = \lambda_I$ of monoidal categories [27, Equation 5.2], which recognizes that left and right unitors become equivalent at the monoidal unit I . This yields equivalences as follows:

$$\begin{aligned} \Phi_\rho^n(a *_0 \text{id}_x^n) &: \Theta_\rho^n(a *_0 \text{id}_x^n) \rightarrow \Theta_{n-1,0}^n(a *_0 \text{id}_x^n) \\ \Phi_\lambda^n(\text{id}_x^n *_0 b) &: \Theta_\lambda^n(\text{id}_x^n *_0 b) \rightarrow \Theta_{n-1,0}^n(\text{id}_x^n *_0 b) \end{aligned}$$

which comprise our next cell in the composite:

$$\begin{aligned} X_2 &= \Phi_\rho^n(a *_0 \text{id}_x^n) *_{n-1} \Phi_\lambda^n(\text{id}_x^n *_0 b) \\ &: \Theta_\rho^n(a *_0 \text{id}_x^n) *_{n-1} \Theta_\lambda^n(\text{id}_x^n *_0 b) \rightarrow \Theta_{n-1,0}^n(a *_0 \text{id}_x^n) *_{n-1} \Theta_{n-1,0}^n(\text{id}_x^n *_0 b) \end{aligned}$$

Step 3. Pseudofunctoriality of Padding. At this point, both $a *_0 \text{id}_x^n$ and $\text{id}_x^n *_0 b$ are present in the term with the same unbiased padding. This allows us to apply a *pseudofunctoriality of padding* construction, which relates the composite of

padding to the padding of the composite. This gives us the following:

$$\begin{aligned} X_3 &= \Xi_{n-1,0}^n(a *_0 \text{id}^n x, \text{id}_x^n *_0 b) \\ &: \Theta_{n-1,0}^n(a *_0 \text{id}_x^n) *_0 *_{n-1} \Theta_{n-1,0}^n(\text{id}_x^n *_0 b) \rightarrow \Theta_{n-1,0}^n((a *_0 \text{id}_x^n) *_0 *_{n-1} (\text{id}_x^n *_0 b)) \end{aligned}$$

Step 4. Interchanger. In the final step, we apply the standard interchanger coherence, which recognizes that the following composites are equivalent:

$$\zeta^n : (a *_0 \text{id}_x^n) *_0 *_{n-1} (\text{id}_x^n *_0 b) \rightarrow a *_0 b$$

We apply an unbiased padding to this interchanger to obtain the final cell in our composite:

$$X_4 : \Theta_{n-1,0}^n((a *_0 \text{id}_x^n) *_0 *_{n-1} (\text{id}_x^n *_0 b)) \rightarrow \Theta_{n-1,0}^n(a *_0 b)$$

This concludes our construction. The construction of $H_{0,n-1}^n$ makes use of similar components.

Acknowledgements

We would like to thank Alex Corner, Eric Finster and Alex Rice for helpful conversations.

2 CaTT and Previous Work

The type theory **CaTT** gives a convenient inductive syntax for the theory of ω -categories. More specifically, contexts Γ of **CaTT** correspond to *finite computads*, which are finite generating data for free ω -categories. Terms $\Gamma \vdash t : A$ of Γ correspond to the cells of the ω -category generated by Γ , with the type A indicating the source, target, and dimension of t . Models of **CaTT** in the sense of Dybjer [18] are exactly ω -categories in the sense of Grothendieck-Maltsiniotis and Batanin-Leinster [3, 11, 14], and thus all our constructions hold for any cells with degenerate boundary in an ω -category.

2.1 The Type Theory CaTT

The type theory **CaTT** has 4 kinds of syntactic entities: contexts, types, terms, and substitutions. The introduction rules for this untyped syntax is presented in Fig. 6. We assume a countable alphabet $x, y, z, f, g, h, a, b \dots$ of variable symbols \mathcal{V} . For a context Γ , $\text{Var}(\Gamma)$ is the set of variables in the context.

We give the definition of substitution application and composition in Fig. 7. We define also the *support* $\text{supp}_\Gamma(t)$ (resp. $\text{supp}_\Gamma(A)$, $\text{supp}_\Gamma(\sigma)$) of a term t , (resp. a type A , or substitution σ) relative to a context Γ , to be the union ranging over $(x : B) \in \Gamma$ such that x appears in t (resp. A , σ) of the sets $\{x\} \cup \text{supp}_\Gamma(B)$. When there is no ambiguity, we omit the index Γ . We say

$$\begin{array}{c}
\frac{}{\emptyset : \text{Ctx}} \\
\frac{}{\star : \text{Ty}} \\
\frac{}{x : \text{Tm}} \\
\frac{}{\langle \rangle : \text{Sub}}
\end{array}
\qquad
\frac{\Gamma : \text{Ctx} \quad A : \text{Ty} \quad (x \notin \text{Var}(\Gamma))}{(\Gamma, x : A) : \text{Ctx}}
\qquad
\frac{A : \text{Ty} \quad u : \text{Tm} \quad v : \text{Tm}}{u \rightarrow_A v : \text{Ty}}
\qquad
\frac{\Gamma : \text{Ctx} \quad A : \text{Ty} \quad \sigma : \text{Sub}}{\text{coh}(\Gamma : A)[\sigma] : \text{Tm}}
\qquad
\frac{\sigma : \text{Sub} \quad t : \text{Tm}}{\langle \sigma, x \mapsto t \rangle : \text{Sub}}$$

Figure 6: The untyped syntax of CaTT.

$$\begin{array}{ll}
x[\langle \rangle] := x & x[\langle \sigma, y \mapsto t \rangle] := x[\sigma] \text{ if } x \neq y \\
x[\langle \sigma, x \mapsto t \rangle] := t & \text{coh}(\Gamma : A)[\tau][\sigma] := \text{coh}(\Gamma : A)[\tau \circ \sigma] \\
\star[\sigma] := \star & (u \rightarrow_A v)[\sigma] := u[\sigma] \rightarrow_{A[\sigma]} v[\sigma] \\
\langle \rangle \circ \sigma := \langle \rangle & \langle \tau, x \mapsto t \rangle \circ \sigma := \langle \tau \circ \sigma, x \mapsto t[\sigma] \rangle
\end{array}$$

Figure 7: Definition of the action of substitutions

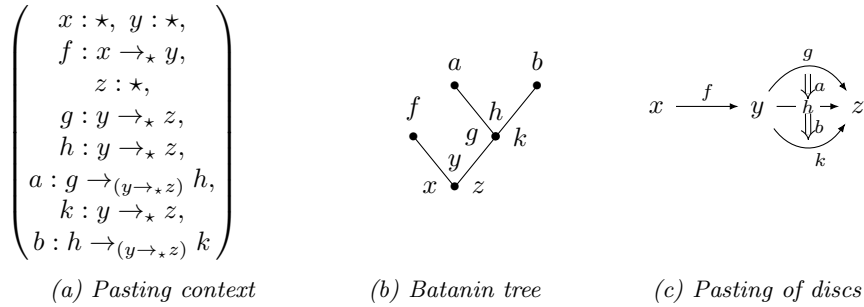


Figure 8: The correspondence between pasting contexts, Bataniin trees and pastings of discs.

a term t (resp. a type A) is *full* in a context Γ if $\text{supp}_\Gamma(t) = \text{Var}(\Gamma)$ (resp. $\text{supp}_\Gamma(A) = \text{Var}(\Gamma)$).

A special role in CaTT is played by *pasting contexts*, a class of contexts that correspond to certain pastings of discs. They may be characterised formally via a bijection with finite rooted planar trees (*Batanin trees*). Given such a tree, we assign to each node N a sequence of labels v_1, \dots, v_{n+1} taken from our set of variables \mathcal{V} , where n is the number of children of N . To each node N we then recursively assign a type $\text{Ty}(N)$ for its variables: the type of the root is \star and the type of the i^{th} child of a node N labelled v_1, \dots, v_n is $(v_i \rightarrow_{\text{Ty}(N)} v_{i+1})$. To produce the context associated with such a labelled tree, it remains to choose an order on the variables: we fix an arbitrary ordering for each tree, such that every positive-dimensional variable is listed after its source and target. We illustrate this correspondence in Fig. 8; formal treatments are available in the literature [10, Theorem 5.5].

We write $\Gamma \vdash_{\text{ps}}$ if Γ is a pasting context. A nontrivial² pasting context Γ has well-defined *source* and *target* pasting contexts $\partial^-\Gamma$ and $\partial^+\Gamma$, corresponding to the trees obtained by removing the leaves of maximal height, and keeping either the leftmost or rightmost labels for the new leaves, respectively. Denoting Γ the pasting context defined in Fig. 8, its source of and targets are given by:

$$\begin{aligned}\partial^-\Gamma &= (x : \star, y : \star, f : x \rightarrow_\star y, z : \star, g : y \rightarrow_\star z) \\ \partial^+\Gamma &= (x : \star, y : \star, f : x \rightarrow_\star y, z : \star, k : y \rightarrow_\star z)\end{aligned}$$

The untyped syntax of CaTT is subject to 4 judgements, the derivation rules for which are presented in Fig. 9:

- $\Gamma \vdash$ the judgement that Γ is a valid context
- $\Gamma \vdash A$ the judgement that A is a valid type in Γ
- $\Gamma \vdash t : A$ the judgement that t is a term of type A in Γ
- $\Gamma \vdash \sigma : \Delta$ the judgement that σ is a substitution $\Gamma \rightarrow \Delta$

Substitution and composition preserve those judgements as expected [6, Prop. 3], and together with the identity substitutions id_Γ , contexts of CaTT thus form a category.

The coh-introduction rule has a side condition (*) which can be satisfied in two ways:

$$(*) \begin{cases} A = u \rightarrow_B v \text{ where } u \text{ is full in } \partial^-\Gamma \text{ and } v \text{ full in } \partial^+\Gamma \\ A = u \rightarrow_B v \text{ where both } u \text{ and } v \text{ are full in } \Gamma \end{cases}$$

In the first case, u and v are ways to compose the source and target of Γ , and the term $\text{coh}(\Gamma : A)[\text{id}]$ is a composition of the entire pasting context Γ . In the second case, $\text{coh}(\Gamma : A)[\text{id}]$ is an equivalence between the operations u and v .

²The trivial pasting context is the point context $\mathbb{P} = (x : \star)$.

$$\begin{array}{c}
\frac{}{\emptyset \vdash} \qquad \frac{\Gamma \vdash \quad \Gamma \vdash A}{(\Gamma, x : A) \vdash} (x \notin \text{Var}(\Gamma)) \\
\\
\frac{}{\Gamma \vdash \star} \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash u : A \quad \Gamma \vdash v : A}{\Gamma \vdash u \rightarrow_A v} \\
\\
\frac{\Gamma \vdash \quad (x : A) \in \Gamma}{\Gamma \vdash x : A} \quad \frac{\Gamma \vdash_{ps} \quad \Gamma \vdash A \quad \Delta \vdash \sigma : \Gamma}{\Delta \vdash \text{coh}(\Gamma : A)[\sigma] : A[[\sigma]]} (*) \\
\\
\frac{}{\langle \rangle : \emptyset \rightarrow \Gamma} \quad \frac{\Gamma \vdash \sigma : \Delta \quad \Delta, x : A \vdash \quad \Gamma \vdash t : A[[\sigma]]}{\Gamma \vdash \langle \sigma, x \mapsto t \rangle : (\Delta, x : A)}
\end{array}$$

Figure 9: Typing rules of CaTT.

Definition 2.1. We say a well-typed term t of the form $\text{coh}(\Gamma : A)[\sigma]$ is a *composite* if its typing derivation uses the first side-condition, and a *coherence* if it uses the second side-condition.

Definition 2.2. We define the dimension of types, terms in a context, and contexts as follows:

$$\begin{aligned}
\dim(\star) &:= -1 & \dim(u \rightarrow_A v) &:= \dim A + 1 \\
\dim_{\Gamma}(t) &:= \dim A + 1 & \text{for } \Gamma \vdash t : A \\
\dim(\Gamma) &:= \max\{\dim_{\Gamma}(x) : x \in \text{Var}(\Gamma)\}
\end{aligned}$$

We write just $\dim(t)$ for $\dim_{\Gamma}(t)$ and $u \rightarrow v$ for $u \rightarrow_A v$ where there is no ambiguity. We refer to terms in a context Γ as *cells* of Γ , and n -dimensional terms as *n-cells*. If $\Gamma \vdash t : u \rightarrow_A v$, we write $u = \partial^- t$ and $v = \partial^+ t$. These source and target operations can be iterated, and we write $\partial_k^- t$ and $\partial_k^+ t$ for the k -dimensional source and target of t , respectively.

The rules of CaTT allow us to construct familiar categorical operations. We first define the *sphere* and *disc* contexts:

Definition 2.3 (Sphere and Disc contexts). We define the contexts \mathbb{S}^{n-1} and \mathbb{D}^n and the type $\mathbb{S}^{n-1} \vdash \mathbb{S}^{n-1}$ for $n \geq 0$:

$$\begin{aligned}
\mathbb{S}^{-1} &:= \emptyset & \mathbb{S}^{n+1} &:= (\mathbb{S}^n, d_-^{n+1} : \mathbb{S}^n, d_+^{n+1} : \mathbb{S}^n) \\
\mathbb{S}^{-1} &:= \star & \mathbb{S}^{n+1} &:= d_-^{n+1} \rightarrow_{\mathbb{S}^n} d_+^{n+1} \\
\mathbb{D}^n &:= (\mathbb{S}^{n-1}, d^n : \mathbb{S}^{n-1})
\end{aligned}$$

Given a context, its *locally-maximal variables* are those which do not appear in the type of any other variable. By type inference and unification, a substitution $\Gamma \vdash \sigma : \Delta$ is fully determined by its action on locally-maximal variables of

Δ . If t_1, \dots, t_n are the images of the locally-maximal variables of Δ under σ , we often use the shorthand $u[[t_1, \dots, t_n]] := u[[\sigma]]$.

Definition 2.4. Give a well-typed term a of dimension n in context Γ , we define its *identity* id_a and *iterated identities* id_a^k by induction as follows:

$$\begin{aligned} \text{id}_a &:= \text{coh}(\mathbb{D}^n : d^n \rightarrow d^n)[a] \\ \text{id}_a^{k+1} &:= \text{id}_{\text{id}_a^k} \end{aligned}$$

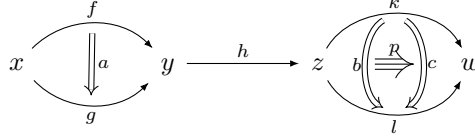
Definition 2.5. Given a pasting context Γ , we define recursively a term comp_Γ , called its *composite*, as follows:

$$\text{comp}_\Gamma := \begin{cases} d^n & \Gamma = \mathbb{D}^n \\ \text{coh}(\Gamma : \text{comp}_{\partial-\Gamma} \rightarrow \text{comp}_{\partial+\Gamma})[\text{id}_\Gamma] & \text{otherwise} \end{cases}$$

We will also use the following more familiar notation for composites and whiskering:

$$t_1 *_k \dots *_k t_n := \text{comp}_\Gamma[[t_1, \dots, t_n]]$$

in the case where Γ is the pasting context obtained from a sequence of discs, potentially of different dimensions, by identifying the variable d_+^k of each disc with the variable d_-^k of its successor. For instance, the whiskering of a 2-cell a , a 1-cell h , and a 3-cell p is denoted by $a *_0 h *_0 p := \text{comp}_\Gamma[[a, h, p]]$. The pasting context over which is defined is illustrated below:



We extend the composition operation $t *_k \dots *_k t$ to the case $\dim(t_i) \leq k$, by adopting the convention that in this case t_1, \dots, t_n are composable only if they are all equal, in which case $t *_k \dots *_k t := t$. With this convention, for any $n, k \in \mathbb{N}$, we have

$$\partial^\pm(\text{id}_x^n *_k \text{id}_x^n) = \text{id}_x^{n-1} *_k \text{id}_x^{n-1}$$

In weak ω -categories, those compositions are not strictly associative nor unital. However, the second side condition of the coh-introduction rule allows us to construct, for example, unitors:

$$\begin{aligned} u_x &:= \text{coh}((x : \star) : \text{id}_x \rightarrow \text{id}_x *_0 \text{id}_x)[x] \\ \rho_f &:= \text{coh}((x, y : \star, f : x \rightarrow y) : f *_0 \text{id}_y \rightarrow f)[f] \end{aligned}$$

Definition 2.6. Throughout this article, we will use the notation $\mathbb{P} := (x : \star)$ for the point context with the variable named x . In this context, we also define the following terms and type, which play a fundamental role in the construction of the cells $\mathbb{H}_{k,l}^n$:

$$(\text{id}_x^n)^{*l} := (\text{id}_x^n)^{*l} (\text{id}_x^n) \quad I_k^n := (\text{id}_x^n)^{*k} \rightarrow (\text{id}_x^n)^{*k}$$

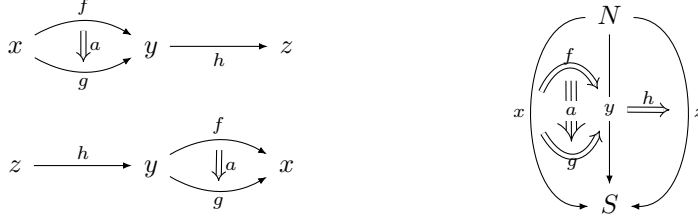


Figure 10: A context (top left), its suspension (right), and its $\{1\}$ -opposite (bottom left).

2.2 Meta-Operations

Various meta-operations have been introduced [6, 8, 9, 12] for CaTT allowing for the automatic construction of complex terms. We give a concise presentation of some of these that we will leverage below.

Suspension. This meta-operation was defined and implemented for CaTT by Benjamin [6, Sec. 3.2], and analogous to the suspension from topology. Suspending a context Γ produces another context $\Sigma\Gamma$ comprised of two new 0-dimensional variables N, S , as well as all variables of Γ . A variable x of type A in context Γ has type ΣA , obtained by formally replacing the base type \star with the type $N \rightarrow S$, in context $\Sigma\Gamma$. This increases by 1 the dimension of the variables, as illustrated in Fig. 10.

Definition 2.7. The suspension meta-operation is defined on the syntax of CaTT as follows:

$$\begin{aligned}
 \Sigma\emptyset &:= (N : \star, S : \star) & \Sigma(\Gamma, x : A) &:= (\Sigma\Gamma, x : \Sigma A) \\
 \Sigma\star &:= N \rightarrow_{\star} S & \Sigma(u \rightarrow_A v) &:= \Sigma u \rightarrow_{\Sigma A} \Sigma v \\
 \Sigma x &:= x & \Sigma(\text{coh}(\Gamma : A)[\sigma]) &:= \text{coh}(\Sigma\Gamma : \Sigma A)[\Sigma\sigma] \\
 \Sigma\langle \rangle &:= \langle N \mapsto N, S \mapsto S \rangle & \Sigma\langle \sigma, x \mapsto t \rangle &:= \langle \Sigma\sigma, x \mapsto \Sigma t \rangle
 \end{aligned}$$

Opposites. Opposites for weak-categories have been studied by Benjamin and Markakis [8]. Whereas a 1-category has a single opposite, ω -categories have opposites for each subset $M \subseteq \mathbb{N}_{>0}$, corresponding to flipping the direction of cells of dimension $n \in M$.

Definition 2.8. For $M \subseteq \mathbb{N}_{>0}$, the opposite meta-operation op_M is defined

on the syntax of CaTT as follows:

$$\begin{aligned}
(\emptyset)^{\text{op } M} &:= \emptyset & (\Gamma, x : A)^{\text{op } M} &:= (\Gamma^{\text{op } M}, x : A^{\text{op } M}) \\
(\star)^{\text{op } M} &:= \star & x^{\text{op } M} &:= x & \langle \rangle^{\text{op } M} &:= \langle \rangle \\
(u \rightarrow_A v)^{\text{op } M} &:= \begin{cases} u^{\text{op } M} \rightarrow_{A^{\text{op } M}} v^{\text{op } M} & \dim u + 1 \notin M \\ v^{\text{op } M} \rightarrow_{A^{\text{op } M}} u^{\text{op } M} & \dim u + 1 \in M \end{cases} \\
(\text{coh}(\Gamma : A)[\sigma])^{\text{op } M} &:= \text{coh}(\Gamma' : A^{\text{op } M} \llbracket \gamma \rrbracket)[\gamma^{-1} \circ \sigma^{\text{op } M}] \\
\langle \sigma, x \mapsto t \rangle^{\text{op } M} &:= \langle \sigma^{\text{op } M}, x^{\text{op } M} \mapsto t^{\text{op } M} \rangle
\end{aligned}$$

Where Γ' is uniquely determined as the pasting context isomorphic to $\Gamma^{\text{op } M}$ under a unique isomorphism $\gamma : \Gamma^{\text{op } M} \rightarrow \Gamma'$ which reorders the variables. When $M = \{1\}$, we write $(-)^{\text{op}}$ for $(-)^{\text{op } M}$. This construction is illustrated in Fig. 10.

Chosen inverses. An n -cell $f : x \rightarrow y$ in an ω -is coinductively defined to be an *equivalence* [15] if there is an n -cell $g : y \rightarrow x$, together with two equivalences

$$\varepsilon : f *_{n-1} g \rightarrow \text{id}_x \qquad \eta : \text{id}_y \rightarrow g *_{n-1} f$$

When this is the case, we say that g is an *inverse* of f . Benjamin and Markakis [9] have shown that in CaTT , all coherences are equivalences, and all composites $t = \text{coh}(\Gamma : A)[\sigma]$ where σ maps all maximal-dimension variables to equivalences are equivalences. For such equivalences $t : u \rightarrow v$, the authors construct a chosen inverse, denoted t^{-1} , and cancellators ε and η . They also prove the following result

Lemma 2.9. *Every term $\Gamma \vdash t : A$ with $\dim(t) > \dim(\Gamma)$ is an equivalence.*

We extend the notion of equivalence to that of congruence between terms of CaTT . This notion is more generic insofar that it allows for two cells with different types to be congruent. The cells $H_{k,l}^n$ that we will construct in this paper are congruences.

Definition 2.10. The *congruence* is the smallest equivalence relation such that an n -cell is congruent to its composite in any dimension with a coherence, and equivalent cells are congruent.

Functorialisation and Naturality. Composites in ω -categories are functorial with respect to their arguments, while coherences are natural. This is made precise by the *functorialisation* [6, §3.4] and the naturality meta-operations [12]. Both operations can be seen as the *depth-0* and *depth-1* cases of the same inductive scheme described below. Here, the *depth* is a parameter defined for contexts Γ , types $\Gamma \vdash A$, terms $\Gamma \vdash t : A$ and substitutions $\Gamma \vdash \sigma : \Delta$, and for a

set of variables $X \subseteq \text{Var } \Gamma$ by

$$\begin{aligned} \text{depth}_X t &= \max\{\dim t - \dim x : x \in \text{supp}(t) \cap X\} \\ \text{depth}_X A &= \max\{\dim A - \dim x : x \in \text{supp}(A) \cap X\} \\ \text{depth}_X \sigma &= \max\{\text{depth}_X x[\sigma] : x \in \text{Var } \Delta\} \\ \text{depth}_X \Gamma &= \text{depth}_X(\text{id}_\Gamma) \end{aligned}$$

where $\max \emptyset = -1$. The scheme further requires that the set X is *up-closed*, meaning that if a variable $x \in X$ appears in the support of some variable $y \in \text{Var}(\Gamma)$, then also $y \in X$. To present the definition, we introduce the preimage X_σ of a set of variables X under a substitution $\Gamma \vdash \sigma : \Delta$ as follows:

$$X_\sigma = \{y \in \text{Var}(\Delta) : \text{supp}(y[\sigma]) \cap X \neq \emptyset\}$$

The construction proceeds recursively on the derivation tree to produce for every context $\Gamma \vdash$ and every up-closed $X \subseteq \text{Var}(\Gamma)$ such that $\text{depth}_X \Gamma \leq 1$, a new context $\Gamma \uparrow X$ together with substitutions:

$$\Gamma \uparrow X \vdash \text{in}^\pm : \Gamma$$

Moreover, it produces for every term $\Gamma \vdash t : A$ such that $0 \leq \text{depth}_X t \leq 1$, a new term:

$$\Gamma \uparrow X \vdash t \uparrow X : A \uparrow^t X$$

and for every substitution $\Gamma \vdash \sigma : \Delta$ such that $\text{depth}_X \sigma \leq 1$, a new substitution:

$$\Gamma \uparrow X \vdash \sigma \uparrow X : \Delta \uparrow X_\sigma$$

When $\Gamma \vdash_{\text{ps}}$ is a pasting context, it also produces for full types $\Gamma \vdash A$ such that $0 \leq \text{depth}_X(\text{coh}(\Gamma : A)[\text{id}]) \leq 1$, a term:

$$\Gamma \uparrow X \vdash \text{coh}(\Gamma : A) \uparrow X : A \uparrow^{\text{coh}(\Gamma:A)[\text{id}]} X$$

- *Contexts.* This procedure duplicates the variables in X and adds a connecting variable relating the two copies. More formally, it is given by:

$$\emptyset \uparrow \emptyset := \emptyset$$

If $x \notin X$, we define:

$$(\Gamma, x : A) \uparrow X := (\Gamma \uparrow X, x : A)$$

If $x \in X$, we let $X' = X \setminus \{x\}$ and define:

$$(\Gamma, x : A) \uparrow X := (\Gamma \uparrow X', x^- : A, x^+ : A, \vec{x} : A \uparrow^x X)$$

The inclusion substitutions in^\pm are determined by:

$$y[\text{in}^\pm] = \begin{cases} y & \text{if } y \notin X \\ y^\pm & \text{if } y \in X \end{cases}$$

- *Types.* The type $A \uparrow^t X$ relates the terms $t[[\text{in}^-]]$ and $t[[\text{in}^+]]$. It is an arrow type of the form $L_{A,t,X} \rightarrow R_{A,t,X}$, where if $A = \star$ we define $L_{A,t,X}$ as $t[[\text{in}^-]]$ and $R_{A,t,X}$ as $t[[\text{in}^+]]$, and when $A = u \rightarrow v$ they are given by:

$$L_{A,t,X} := \begin{cases} t[[\text{in}^-]] *_{n-1} (v \uparrow X) & \text{if } \text{supp}(v) \cap X \neq \emptyset \\ t[[\text{in}^-]] & \text{otherwise} \end{cases}$$

$$R_{A,t,X} := \begin{cases} (u \uparrow X) *_{n-1} t[[\text{in}^+]] & \text{if } \text{supp}(u) \cap X \neq \emptyset \\ t[[\text{in}^+]] & \text{otherwise} \end{cases}$$

- *Terms.* The term $t \uparrow X$ is defined recursively by:

$$x \uparrow X := \vec{x}$$

$$\text{coh}(\Delta : A)[\sigma] \uparrow X := (\text{coh}(\Delta : A) \uparrow X_\sigma)[\sigma \uparrow X]$$

- *Substitutions.* For $x \notin X_\sigma$, $\sigma \uparrow X$ is given by:

$$x[\sigma \uparrow X] := x[\sigma]$$

and for $x \in X_\sigma$ by:

$$x^\pm[\sigma \uparrow X] := x[\sigma \circ \text{in}^\pm]$$

$$\vec{x}[\sigma \uparrow X] := x[\sigma] \uparrow X$$

- *Full types.* Let $t = \text{coh}(\Gamma : A)[\text{id}]$. When $\text{depth}_X(\Gamma) = 0$, then $\Gamma \uparrow X$ is again a pasting diagram and $A \uparrow^t X$ is full, allowing one to define

$$\text{coh}(\Gamma : A) \uparrow X = \text{coh}(\Gamma \uparrow X : A \uparrow^t X)[\text{id}]$$

When $\text{depth}_X(\Gamma) = 1$, then $\Gamma \uparrow X$ is no longer a pasting diagram, and t is a composite. The term $\text{coh}(\Gamma : A) \uparrow X$ has been constructed by Benjamin et al. [12] in this case.

Example 2.11. The functorialisation of the composite $f *_0 g$ of two 1-cells with respect to f is the whiskering $\text{comp}_\Gamma[\vec{f}, g]$ where Γ is the following pasting context:

$$\begin{array}{ccc} & f & \\ x & \xrightarrow{\quad} & y \xrightarrow{h} z \\ & \Downarrow a & \\ & g & \end{array}$$

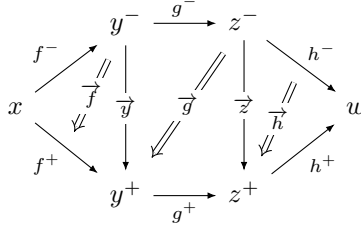
Example 2.12. Consider the right unitor $\rho_f : f *_0 \text{id}_y \rightarrow f$ in the context $\Gamma_f = (x, y : \star, f : x \rightarrow_\star y)$. Letting $X = \{f\}$, we get a term $\rho_f \uparrow X$ filling the following square:

$$\begin{array}{ccc} f^- *_0 \text{id}_y & \xrightarrow{\rho_{f^-}} & f^- \\ \vec{f} *_0 \text{id}_y \downarrow & \xrightarrow{\rho_{\vec{f}}} & \downarrow \vec{f} \\ f^+ *_0 \text{id}_y & \xrightarrow{\rho_{f^+}} & f^+ \end{array}$$

Example 2.13. Consider the context:

$$\mathbf{3} := (x, y, z, w : *, f : x \rightarrow y, g : y \rightarrow z, h : z \rightarrow w)$$

Letting $X = \{f, y, g, z, h\}$ the context $\mathbf{3} \uparrow X$ is given by:



Consider now the term $t = f *_0 g *_0 h$ over $\mathbf{3}$. Its naturality with respect to X is a term over $\mathbf{3} \uparrow X$ of type:

$$f^- *_0 g^- *_0 h^- \rightarrow f^+ *_0 g^+ *_0 h^+$$

Given 2-cells a, b, c whose boundaries match as in the context $\mathbf{3} \uparrow X$, we define their *hexagonal composite*:

$$\text{hexcomp}[[a, b, c]] := (t \uparrow X)[[a, b, c]]$$

Let σ be a substitution whose target is an iterated suspension of the context of the hexagonal composition:

$$\sigma : \Gamma \rightarrow \Sigma^k(\mathbf{3} \uparrow X)$$

Denoting a, b and c the respective images of \vec{f}, \vec{g} and \vec{h} under the action of σ , we use suspension implicitly and write:

$$\text{hexcomp}[[a, b, c]] := (\Sigma^k(t \uparrow X))[[\sigma]]$$

We will use the hexagonal composition and its suspensions in for construction of the repadding in Sec. 3.

3 Padding and Repadding

This section is dedicated to the presentation of our theory of padding, which lies at the heart of our method to construct congruences. Our technique is inspired by the padding constructions in Fujii et al. [22] and Finster et al. [21], but takes a more axiomatic approach, describing the general shape of such paddings.

3.1 Padding

Our method for padding cells involves recursively adjusting their boundaries as necessary, starting with the lowest dimension where they differ, proceeding until the cell has the desired type. To capture this dimensionwise recursive structure, we introduce a notion of filtration.

Definition 3.1. A *filtration* $\Gamma = (\Gamma^i, v^i, \sigma^i)_{i=m}^n$ of *height* m constitutes a sequence of contexts Γ^i of dimension i , together with a chosen variable v^i in context Γ^i and a sequence of substitutions σ^i for $m < i \leq n$ satisfying:

$$\Gamma^i \vdash \sigma^i : (\Gamma^{i-1} \uparrow v^{i-1}) \quad \overrightarrow{v^i} \llbracket \sigma^{i+1} \rrbracket = v^{i+1}$$

A family of types $\mathbf{A} = (A^i)_{i=m}^n$ is *adapted* to the filtration Γ when $\Gamma^m \vdash v^m : A^m$, and for all $i \in \{m+1, \dots, n\}$, there exist terms s^i, t^i satisfying:

$$\begin{aligned} \Gamma \vdash s^i : A^i \llbracket \sigma^{i+1} \rrbracket \quad \Gamma \vdash t^i : A \llbracket \sigma^{i+1} \rrbracket \\ A^{i+1} = s^i \rightarrow_{A^i \llbracket \sigma^{i+1} \rrbracket} t^i \end{aligned}$$

Finally a set of *padding data* $\mathbf{p} = (p^i, q^i)_{i=m}^{n-1}$ for the type family \mathbf{A} adapted to the filtration Γ is defined mutually inductively together with its *associated padding* $\Theta_{\mathbf{p}}$. Padding data consists in a family of terms p^i and q^i satisfying:

$$\begin{aligned} \Gamma^{i+1} \vdash p^i : s^i \rightarrow \Theta_{\mathbf{p}}^i \llbracket \text{in}^- \circ \sigma^{i+1} \rrbracket \\ \Gamma^{i+1} \vdash q^i : \Theta_{\mathbf{p}}^i \llbracket \text{in}^+ \circ \sigma^{i+1} \rrbracket \rightarrow t^i \\ v^{i-1} \notin \text{supp}(p^i) \cup \text{supp}(q^i) \end{aligned}$$

Its associated padding is a term $\Gamma^i \vdash \Theta_i^i : A^i$ defined for $m \leq i \leq n$ by:

$$\begin{aligned} \Theta_{\mathbf{p}}^m &:= v^m \\ \Theta_{\mathbf{p}}^{i+1} &:= p^i *_i (\Theta_{\mathbf{p}}^i \uparrow v^i) \llbracket \sigma^{i+1} \rrbracket *_i q^i \end{aligned} \quad (\dagger)$$

Fig. 11 illustrates the shape of the paddings of height 0 and dimension up to 2. We now define the *unbiased padding*, illustrated in Fig. 12. This is the padding appearing in the type of the Eckmann-Hilton cells $H_{k,l}^n$, as well as in the respective source and targets of X_3 and X_4 in Fig. 5. It transports terms from type I_k^n of Def. 2.6 to type I_l^n using *generalised unbiased unitors*. It will turn out to be self-dual, i.e. invariant under opposites, a crucial property necessary to define the cells $H_{k,l}^n$ and assemble into the commutativity cells $\text{EH}_{k,l}^n$.

Definition 3.2 (Unbiased unitors and unbiased paddings). For $n \geq 2$ and $k, l < n$, we denote $m = \min\{k, l\} + 1$, and we introduce the filtration $\Gamma_{k,l}^n = (\Gamma_l^i, v_l^i, \sigma^i)_{i=m}^n$ where

$$\Gamma_l^i = (x : \star, v_l^i : I_l^{i-1}) \quad \sigma^i = \langle x \mapsto x, v_l^i \mapsto v_l^{i+1} \rangle$$

We then define a set of padding data $\mathbf{u}_{k,l}^n = (p_{k,l}^i, q_{k,l}^i)_{i=m}^n$ for the family $(I_k^i)_{i=m}^n$ adapted to $\Gamma_{k,l}^n$ with associated padding denoted $\Theta_{k,l}^i$, by:

$$\begin{aligned} p_{k,l}^i &:= \text{coh}(\mathbb{P} : (\text{id}_x)^{*k} \rightarrow \Theta_{k,l}^i \llbracket (\text{id}_x)^{*l} \rrbracket) [x] \\ q_{k,l}^i &:= \text{coh}(\mathbb{P} : \Theta_{k,l}^i \llbracket (\text{id}_x)^{*l} \rrbracket \rightarrow (\text{id}_x)^{*k}) [x] = (p_{k,l}^i)^{-1} \end{aligned}$$

We call the terms $p_{k,l}^i$ of this family the *generalised unbiased unitors* and its associated padding $\Theta_{k,l}^i$ the *unbiased padding*.

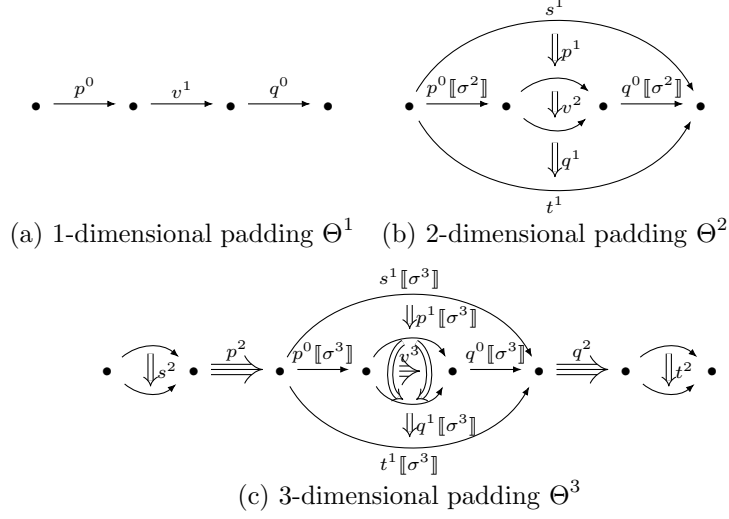


Figure 11: Paddings of height 0 with simplified notation, e.g. writing $p^0[[\sigma^3]]$ for $p^0[\text{in}_{\Gamma_1}^- \circ \sigma^2 \circ \text{in}_{\Gamma_2}^- \circ \sigma^3]$.

Lemma 3.3. *The unbiased padding is self-dual. For any $M \subseteq \mathbb{N}$, we have*

$$(\Theta_{k,l}^i)^{\text{op } M} \llbracket v_l^n \rrbracket = \Theta_{k,l}^i$$

Proof. Appendix, Lemma D.2 □

We now introduce the *generalised biased unitors* and their associated *biased paddings*. These are illustrated in Fig. 12 and play a key role in our construction of the cells $H_{n-1,0}^n$, appearing in X_1 of Fig. 5. To shorten the construction, we leverage a duality argument, allowing us to focus only on right unitors. In fact, we define two flavours $\rho^n, \tilde{\rho}^n$ of right unitors and respective associated padding $\Theta_\rho^n, \Theta_{\tilde{\rho}}^n$, the first appearing in the construction of $H_{n-1,0}^n$ and the latter in that of $H_{0,n-1}^n$.

Definition 3.4 (Generalised unitors and biased paddings). We define the filtration $\Gamma_\rho^n = (\Gamma_\rho^i, v^i, \sigma_\rho^i)$ by:

$$\begin{aligned} \Gamma_\rho^i &= (\mathbb{S}^i, v^i : d_-^{i-1} *_0 \text{id}^{i-1}(d_+^0) \rightarrow d_+^{i-1} *_0 \text{id}^{i-1}(d_+^0)) \\ \sigma_\rho^i &= \langle d_\pm^j \mapsto d_\pm^j, (v^{i-1})^\pm \mapsto \overrightarrow{d_\pm^{i-1}}, (v^{i-1}) \mapsto v^i \rangle \end{aligned}$$

We then define the padding data $\mathbf{p}_\rho^n = (p_\rho^i, q_\rho^i)_{i=1}^{n-1}$ for the type family $(d_-^{i-1} \rightarrow d_+^{i-1})_{i=1}^n$ adapted to Γ_ρ^n , whose associated padding we denote Θ_ρ^n , as follows:

$$\begin{aligned} \rho^i &:= \text{coh}(\mathbb{D}^i : \Theta_\rho^i \llbracket d^i *_0 \text{id}_{d_+^0} \rrbracket \rightarrow d^i) [\text{id}_{\mathbb{D}^i}] \\ p_\rho^i &:= (\rho^i)^{-1} \llbracket d_-^i \rrbracket & q_\rho^i &:= \rho^i \llbracket d_+^i \rrbracket \end{aligned}$$

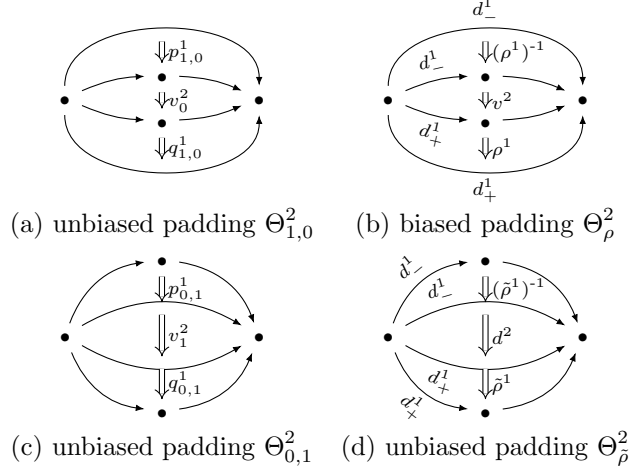


Figure 12: Biased and unbiased paddings in dimension 2. The unlabelled arrows are identities.

We also define the filtration $\mathbf{\Gamma}_\rho^n = (\mathbb{D}^i, d^i, \sigma^i)_{i=1}^n$ where \mathbb{D}^i is the i -disc context of Def. 2.3, and σ^i is the isomorphism between $(\mathbb{D}^{i-1} \uparrow d^{i-1})$ and \mathbb{D}^i . Consider the following type family adapted to the filtration $\mathbf{\Gamma}_\rho^n$:

$$(d_-^{i-1} *_0 \text{id}_{d_+^0}^{i-1} \rightarrow d_+^{i-1} *_0 \text{id}_{d_+^0}^{i-1})_{i=1}^n$$

We define padding data $\mathbf{p}_\rho^n = (p_\rho^i, q_\rho^i)_{i=1}^{n-1}$ for this type family, whose associated padding we call Θ_ρ^i , as follows:

$$\begin{aligned} \tilde{\rho}^i &:= \text{coh}(\mathbb{D}^i : \Theta_\rho^i \llbracket d_-^i \rrbracket \rightarrow d^i *_0 \text{id}_{d_+^0}^i) \llbracket \text{id}_{\mathbb{D}^i} \rrbracket \\ p_\rho^i &:= (\tilde{\rho}^i)^{-1} \llbracket d_-^i \rrbracket & q_\rho^i &:= \tilde{\rho}^i \llbracket d_+^i \rrbracket \end{aligned}$$

We call the coherences $\rho^n, \tilde{\rho}^n$ *generalised right unitors*. The paddings Θ_ρ^n and $\Theta_{\tilde{\rho}}^n$ are the *right-biased paddings*. Using the duality, we define the *generalised left unitors* and *left-biased paddings* as follows:

$$\begin{aligned} \lambda^n &:= (\rho^n)^{\text{op}} & \tilde{\lambda}^n &:= (\tilde{\rho}^n)^{\text{op}} \\ \Theta_\lambda^n &:= (\Theta_\rho^n)^{\text{op}} & \Theta_{\tilde{\lambda}}^n &:= (\Theta_{\tilde{\rho}}^n)^{\text{op}} \end{aligned}$$

We now define morphisms of filtrations. Those will allow us to transport paddings over difference filtrations. Using such morphisms, we can transport the left-biased and right-biased paddings to the filtration of the unbiased padding, and subsequently to relate them in Sec. 3.2. This relation is analogue to the equation $\rho_{\text{id}} = \lambda_{\text{id}}$ from monoidal categories.

Definition 3.5. A *morphism of filtrations* $\psi = (\psi^i)_{i=m}^n$ between filtrations $(\Delta^i, w^i, \tau^i)_{i=m}^n$ and $(\Gamma^i, v^i, \sigma^i)_{i=m}^n$ consists of substitutions $\psi^i : \Delta^i \rightarrow \Gamma^i$ such that $\{w^i\}_{\psi^i} = \{v^i\}$, and the following commutes for each i :

$$\begin{array}{ccc} \Delta^{i+1} & \xrightarrow{\tau^{i+1}} & \Delta^i \uparrow w^i \\ \psi^{i+1} \downarrow & & \downarrow \psi^i \uparrow w^i \\ \Gamma^{i+1} & \xrightarrow{\sigma^{i+1}} & \Gamma^i \uparrow v^i \end{array} \quad (1)$$

Given a family of types $\mathbf{A} = (A^i)_{i=m}^n$, and padding data $\mathbf{p} = (p^i, q^i)_{i=m}^{n-1}$ for \mathbf{A} with associated padding $\Theta_{\mathbf{p}}$, we denote:

$$\begin{aligned} \mathbf{A}[\psi] &:= (A^i[\psi^i])_{i=m}^n & \mathbf{p}[\psi] &:= (p^i[\psi^{i+1}], q^i[\psi^{i+1}])_{i=m}^{n-1} \\ \Theta_{\mathbf{p}}[\psi] &:= (\Theta_{\mathbf{p}}^i[\psi^i])_{i=m}^n \end{aligned}$$

Lemma 3.6. *Given $\psi : \Delta \rightarrow \Gamma$ a morphism of filtrations, if \mathbf{A} is a type family adapted to Γ , then $\mathbf{A}[\psi]$ is adapted to Δ . If \mathbf{p} is padding data for \mathbf{A} with associated padding $\Theta_{\mathbf{p}}$, then $\mathbf{p}[\psi]$ is padding data for $\mathbf{A}[\psi]$, with associated padding then $\Theta_{\mathbf{p}}[\psi]$.*

Proof. See Appendix C. □

We now proceed with the definition of the unbiased paddings of the identity, which are padding data for the same type as $\mathbf{u}_{k,0}^n$ and $\mathbf{u}_{k,n-1}^n$, but distinct from them. In Sec. 3.2, we define the *unbiasing repaddings* to relate these distinct paddings.

Definition 3.7. We define two morphisms of filtrations:

$$\begin{array}{ccc} \psi_{\rho} : \Gamma_{k,0}^n \rightarrow \Gamma_{\rho}^n & & \psi_{\bar{\rho}} : \Gamma_{k,n-1}^n \rightarrow \Gamma_{\bar{\rho}}^n \\ v^i[\psi_{\rho}^i] = v_{\rho}^i & & d^i[\psi_{\bar{\rho}}^i] = v_{n-1}^i \end{array}$$

Applying these morphisms to the biased paddings, we obtain new padding data, that we call the *biased paddings of the identity*

$$\mathbf{p}_{\rho}^n[\psi_{\rho}] \qquad \mathbf{p}_{\bar{\rho}}^n[\psi_{\bar{\rho}}]$$

We conclude this section with a presentation of suspension of paddings. While it does not appear in the construction of the steps presented in Fig. 5, this notion still plays an important role for constructing the cells $H_{k,l}^n$. It is central in Lemma 4.2, allowing us to leverage the suspension meta-operation to construct $H_{k+1,l+1}^{n+1}$ from $H_{k,l}^n$.

Definition 3.8. Let $\Gamma = (\Gamma^i, v^i, \sigma^i)_{i=m}^n$ be a filtration, and $\mathbf{p} = (p^i, q^i)_{i=m}^n$ be padding data. We define:

$$\begin{aligned} \Sigma\Gamma &:= (\Sigma\Gamma^{i-1}, \Sigma v^{i-1}, \Sigma\sigma^i)_{i=m+1}^{n+1} \\ \Sigma\mathbf{p} &:= (\Sigma p^{i-1}, \Sigma q^{i-1})_{i=m+1}^{n+1} \end{aligned}$$

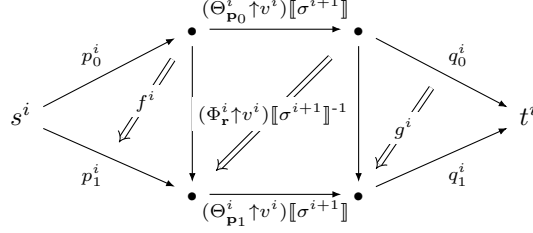


Figure 13: The repadding cell $\Phi_{\mathbf{r}}^{i+1}$.

Lemma 3.9. *If Γ is a filtration, then so is $\Sigma\Gamma$. If the type family \mathbf{A} is adapted to Γ , then $\Sigma\mathbf{A}$ is adapted to $\Sigma\Gamma$, and if \mathbf{p} is padding data for \mathbf{A} , then $\Sigma\mathbf{p}$ is padding data for $\Sigma\mathbf{A}$, with associated padding $\Sigma\Theta_{\mathbf{p}}$:*

Proof. See Appendix C. □

3.2 Repadding

We now introduce repadding, which allows us to change between two padding datas for the same type family. This will constitute the heart of the construction of X_2 in Fig. 5.

Definition 3.10. Consider a filtration $(\Gamma^i, v^i, \sigma^i)_{i=m}^n$, a type family \mathbf{A} adapted to it, and two sets of padding data for \mathbf{A} :

$$\mathbf{p}_0 = (p_0^i, q_0^i)_{i=m}^{n-1} \quad \mathbf{p}_1 = (p_1^i, q_1^i)_{i=m}^{n-1}$$

We define sets of *repadding data* \mathbf{r} and their associated repadding $\Phi_{\mathbf{r}}^i$ together by mutual induction. A set of repadding data $\mathbf{r} = (f^i, g^i)_{i=m}^{n-1}$, constitutes of families of

$$\begin{aligned} \Gamma^{i+1} \vdash f^i : p_0^i *_i \Phi_{\mathbf{r}}^i[\text{in}^- \circ \sigma^{i+1}] &\rightarrow p_1^i \\ \Gamma^{i+1} \vdash g^i : q_0^i &\rightarrow \Phi_{\mathbf{r}}^i[\text{in}^+ \circ \sigma^{i+1}] *_i q_1^i \end{aligned}$$

Its associated repadding the term $\Gamma^i \vdash \Phi_{\mathbf{r}}^i : \Theta_{\mathbf{p}_0}^i \rightarrow \Theta_{\mathbf{p}_1}^i$ defined by:

$$\begin{aligned} \Phi_{\mathbf{r}}^m &:= \text{id}_{v^m} \\ \Phi_{\mathbf{r}}^{i+1} &:= \text{hexcomp}[[f^i, ((\Phi_{\mathbf{r}}^i \uparrow v^i)[[\sigma^{i+1}]]^{-1}, g^i]] \end{aligned}$$

The definition of $\Phi_{\mathbf{r}}^{i+1}$, illustrated in Fig. 13, uses an inverse, which exists by Lemma 2.9. We now introduce the *unbiasing repaddings* as the crucial ingredients of cell X_2 of Fig. 5, allowing us to change a biased padding applied to the identity into an unbiased padding.

Definition 3.11 (Unbiasing repaddings). Given $n \in \mathbb{N}$, we recall the biased paddings of the identity $\mathbf{p}_\rho^n \llbracket \psi_\rho \rrbracket$ from Def. 3.7 and the unbiased paddings $\mathbf{u}_{n-1,0}^n$ from Def. 3.2. We define a set of repadding data $\mathbf{r}_\rho^n = (f_\rho^i, g_\rho^i)_{i=1}^{n-1}$ from $\mathbf{p}_\rho^n \llbracket \psi_\rho \rrbracket$ to $\mathbf{u}_{n-1,0}^n$ and its associated repadding Φ_ρ^i , as follows:

$$\begin{aligned} f_\rho^i &:= \text{coh}(\mathbb{P} : p_\rho^i \llbracket \psi_\rho^i \rrbracket *_i \Phi_\rho^i \llbracket \text{in}^- \circ \sigma^{i+1} \rrbracket \rightarrow p_{n-1,0}^i) [x] \\ g_\rho^i &:= \text{coh}(\mathbb{P} : q_\rho^i \llbracket \psi_\rho^i \rrbracket \rightarrow \Phi_\rho^i \llbracket \text{in}^+ \circ \sigma^{i+1} \rrbracket *_i q_{n-1,0}^i) [x] \end{aligned}$$

Similarly, we define repadding data $\mathbf{r}_{\bar{\rho}}^n = (f_{\bar{\rho}}^i, g_{\bar{\rho}}^i)_{i=1}^{n-1}$ from $\mathbf{u}_{0,n-1}^n$ to $\mathbf{p}_{\bar{\rho}}^n \llbracket \psi_{\bar{\rho}} \rrbracket$, and its associated repadding $\Phi_{\bar{\rho}}^i$, as follows:

$$\begin{aligned} f_{\bar{\rho}}^i &:= \text{coh}(\mathbb{P} : p_{\bar{\rho}}^i \llbracket \psi_{\bar{\rho}}^i \rrbracket *_i \Phi_{\bar{\rho}}^i \llbracket \text{in}^- \circ \sigma^{i+1} \rrbracket \rightarrow p_{0,n-1}^i) [x] \\ g_{\bar{\rho}}^i &:= \text{coh}(\mathbb{P} : q_{\bar{\rho}}^i \llbracket \psi_{\bar{\rho}}^i \rrbracket \rightarrow \Phi_{\bar{\rho}}^i \llbracket \text{in}^+ \circ \sigma^{i+1} \rrbracket *_i q_{0,n-1}^i) [x] \end{aligned}$$

We call the associated repaddings the *right-unbiasing repaddings*. They provide the following equivalences, which are needed for X_2 in Fig. 5:

$$\begin{aligned} \Gamma_0^n \vdash \Phi_\rho^n : \Theta_\rho^n \llbracket v_0^n \rrbracket &\rightarrow \Theta_{n-1,0}^n \\ \Gamma_{n-1}^n \vdash \Phi_{\bar{\rho}}^n : \Theta_{\bar{\rho}}^n \llbracket v_{n-1}^n \rrbracket &\rightarrow \Theta_{0,n-1}^n \end{aligned}$$

We also define the *left-unbiasing repadding*:

$$\Phi_\lambda := (\Phi_\rho)^{\text{op}} \qquad \Phi_{\bar{\lambda}} := (\Phi_{\bar{\rho}})^{\text{op}}$$

By Lemma 3.3, these terms satisfy:

$$\begin{aligned} \Gamma_0^n \vdash \Phi_\lambda^n : \Theta_\lambda^n \llbracket v_0^n \rrbracket &\rightarrow \Theta_{n-1,0}^n \\ \Gamma_{n-1}^n \vdash \Phi_{\bar{\lambda}}^n : \Theta_{\bar{\lambda}}^n \llbracket v_{n-1}^n \rrbracket &\rightarrow \Theta_{0,n-1}^n \end{aligned}$$

3.3 Pseudo-Functoriality of the Unbiased Padding

The key idea for X_3 in Fig. 5 is to construct a witness that relates the unbiased padding of a composite, with the composite of unbiased paddings. We think of this as a pseudo-functoriality property, since it is exactly one of the pieces of data for pseudo-functoriality of a 2-functor.

Proposition 3.12. *In the context $(\Gamma_l^n, w : I_k^{n-1})$ we can construct a cell:*

$$\Xi_{k,l}^n : \Theta_{k,l}^n \llbracket v_l^n \rrbracket *_i \Theta_{k,l}^n \llbracket w \rrbracket \rightarrow \Theta_{k,l}^n \llbracket v_l^n *_i w \rrbracket$$

Proof. Recall the filtration $\mathbf{\Gamma}_{\mathbf{k},1}^n = (\Gamma_l^i, v_l^i, \sigma^i)_{i=m}^n$ for the unbiased padding, defined in Def 3.2. Given a term $\Gamma_l^i \vdash t : A$, we introduce the notation:

$$\begin{aligned} t \uparrow^0 v_l^i &:= t \\ t \uparrow^{k+1} v_l^i &:= ((t \uparrow^k v_l^i) \uparrow v_l^{i+k}) \llbracket \sigma^{i+k+1} \rrbracket \end{aligned}$$

We construct by induction on $i \geq m$ terms $\Xi_{k,l}^{i \uparrow n-i}$ and then specialise this construction to define $\Xi_{k,l}^n := \Xi_{k,l}^{n \uparrow 0}$. The terms $\Xi_{k,l}^{i \uparrow n-i}$ that we construct will be valid in the context Γ_l^n , and will have source:

$$(\Theta_{k,l}^i \uparrow^{n-i} v_l^i) \llbracket v_l^n \rrbracket *_{n-1} (\Theta_{k,l}^i \uparrow^{n-i} v_l^i) \llbracket w \rrbracket$$

and target:

$$(\Theta_{k,l}^i \uparrow^{n-i} v_l^i) \llbracket v_l^n *_{n-1} w \rrbracket$$

When $i = m$, we define $\Xi_{k,l}^{m \uparrow (n-m)} := \text{id}(v_l^m *_{n-1} w)$. When $m < i < n$, we use the equality (Appendix, Lemma C.1):

$$\Theta_{k,l}^i \uparrow^{n-i} v_l^i = p_{k,l}^{i-1} *_{i-1} (\Theta_{k,l}^{i-1} \uparrow^{n-i+1} v_l^{i-1}) *_{i-1} q_{k,l}^{i-1}$$

This allows us define $\Xi_{k,l}^{i \uparrow n-i}$ as a composite of an interchanger with an application of the whiskering of $\Xi_{k,l}^{i-1 \uparrow n-i+1}$. This composite is summarised by the diagram below, where $t = \Theta_{k,l}^{i-1} \uparrow^{n-i+1} v_l^{i-1}$ and the first step is given by an interchanger explicitly constructed in Appendix E.1.

$$\begin{array}{c} \left(p_{k,l}^{i-1} *_{i-1} t \llbracket v_l^n \rrbracket *_{i-1} q_{k,l}^{i-1} \right) *_{n-1} \left(p_{k,l}^{i-1} *_{i-1} t \llbracket w \rrbracket *_{i-1} q_{k,l}^{i-1} \right) \\ \downarrow \text{(interchanger)} \\ p_{k,l}^{i-1} *_{i-1} (t \llbracket v_l^n \rrbracket *_{n-1} t \llbracket w \rrbracket) *_{i-1} q_{k,l}^{i-1} \\ \downarrow \text{whiskering of } \Xi_{k,l}^{i-1 \uparrow n-i+1} \\ p_{k,l}^{i-1} *_{i-1} t \llbracket v_l^n *_{n-1} w \rrbracket *_{i-1} q_{k,l}^{i-1} \end{array}$$

Finally, for $i = n$ we define $\Xi_{k,l}^{n \uparrow 0}$ as the composite of two steps presented below, with $t = \Theta_{k,l}^{n-1} \uparrow v_l^{n-1}$:

$$\begin{array}{c} \left(p_{k,l}^{n-1} *_{n-1} t \llbracket v_l^n \rrbracket *_{n-1} q_{k,l}^{n-1} \right) *_{n-1} \left(p_{k,l}^{n-1} *_{n-1} t \llbracket w \rrbracket *_{n-1} q_{k,l}^{n-1} \right) \\ \downarrow \text{cancellator for the inverses } q_{k,l}^{n-1} \text{ and } p_{k,l}^{n-1} \\ p_{k,l}^{n-1} *_{n-1} (t \llbracket v_l^n \rrbracket *_{n-1} t \llbracket w \rrbracket) *_{n-1} q_{k,l}^{n-1} \\ \downarrow \text{whiskering of } \Xi_{k,l}^{n-1 \uparrow 1} \\ p_{k,l}^{n-1} *_{n-1} t \llbracket v_l^n *_{n-1} w \rrbracket *_{n-1} q_{k,l}^{n-1} \end{array}$$

□

3.4 Iterated Padding

We conclude our section on padding with additional results regarding relating the nested padding of a padding with a padding performed in a single step. This is used in Corollary 4.6 to construct generalisations of the cells $H_{k,l}^n$.

Definition 3.13. Let $\Gamma = (\Gamma^i, v^i, \sigma^i)_{i=m}^n$ be a filtration and B^i a type family adapted to it. Denote A^i the type family defined by $\Gamma^i \vdash v^i : A^i$. Let $\Gamma \setminus v^i$ be the context obtained by removing v^i . Since v^i is locally maximal, this context is well-formed. Moreover, due to the dimension of B^i , we have $\Gamma^i \setminus v^i \vdash B^i$. We define a family $\Gamma_{/B} := (\Gamma_{/B}^i, w^i, \sigma_{/B}^i)_{i=m}^n$ as follows:

$$\begin{aligned} \Gamma_{/B}^i &:= (\Gamma^i \setminus v^i, w^i : B^i) \\ \Gamma_{/B}^i \vdash \sigma_{/B}^i &: \Gamma_{/B}^{i-1} \uparrow w^i \\ x \llbracket \sigma_{/B}^i \rrbracket &:= \begin{cases} w^i & \text{if } x = \overline{w^{i-1}} \\ x \llbracket \sigma^i \rrbracket & \text{if } x \in \Gamma^i \setminus v^i \end{cases} \end{aligned}$$

Proposition 3.14. *For any filtration Γ and type family B adapted to it, the family $\Gamma_{/B}$ is a filtration. Moreover, a type family C is adapted to Γ if and only if it is adapted to $\Gamma_{/B}$.*

Proof. See Appendix F. □

Proposition 3.15. *Given a filtration Γ with two types families B and C adapted to it. Suppose we have padding data $\mathbf{p} = (p_-^i, p_+^i)_{i=m}^{n-1}$ for B adapted to Γ and padding data $\mathbf{q} = (q_-^i, q_+^i)_{i=m}^{n-1}$ for C adapted to $\Gamma_{/B}$. Then there exists padding data $\mathbf{q} \square \mathbf{p} = (q_-^i \boxminus p_-^i, p_+^i \boxplus q_+^i)_{i=m}^{n-1}$ for C adapted to Γ and equivalences:*

$$\Gamma^i \vdash \mu_{\mathbf{q}, \mathbf{p}}^i : \Theta_{\mathbf{q}}^i \llbracket \Theta_{\mathbf{p}}^i \rrbracket \rightarrow \Theta_{\mathbf{q} \square \mathbf{p}}^i$$

Proof. See Appendix F. □

4 Construction of Eckmann-Hilton Cells

We now construct the cells $H_{k,l}^n$ for $0 \leq k, l < n$ with $k \neq l$, using our theory of padding. We first present the construction of $H_{n-1,0}^n$ and $H_{0,n-1}^n$, then describe how suspension allows the construction of $H_{k+1,l+1}^{n+1}$ from $H_{k,l}^n$, while naturality allows the construction of $H_{k,l}^{n+1}$ from $H_{k,l}^n$. This covers all cases. To simplify the notation, we introduce the contexts for these cells and types of these cells:

$$\begin{aligned} \mathbb{E}^n &:= (x : \star, a, b : \text{id}_x^{n-1} \rightarrow \text{id}_x^{n-1}) \\ E_{k,l}^n &:= a *_k b \rightarrow \Theta_{k,l}^n \llbracket a *_l b \rrbracket \end{aligned}$$

One can check that this type is valid in context \mathbb{E}^n . Our main result, Theorem 4.4, gives the construction of cells $H_{k,l}^n$ such that the following judgement is derivable:

$$\mathbb{E}^n \vdash H_{k,l}^n : E_{k,l}^n \tag{2}$$

We begin with the base cases $H_{n-1,0}^n$ and $H_{0,n-1}^n$.

Lemma 4.1. *For every $n \geq 2$, we can construct cells $H_{n-1,0}^n$ and $H_{0,n-1}^n$ satisfying:*

$$\mathbb{E}^n \vdash H_{n-1,0}^n : E_{n-1,0}^n \qquad \mathbb{E}^n \vdash H_{0,n-1}^n : E_{0,n-1}^n$$

Proof. The construction of $H_{n-1,0}^n$ follows exactly the structure in the steps X_1, X_2, X_3, X_4 shown in Fig. 5. We recall the generalised biased unitors (Def. 3.4), unbiasing repaddings (Def. 3.11), and pseudofunctoriality of the unbiased padding (Proposition 3.12):

$$\begin{aligned} \mathbb{D}^n \vdash \rho^n &: \Theta_\rho^n \llbracket d^n *_{\text{id}_{d_+^n}} \rrbracket \rightarrow d^n \\ \mathbb{D}^n \vdash \lambda^n &: \Theta_\rho^n \llbracket \text{id}_{d_+^n} *_{\text{id}_{d_+^n}} d^n \rrbracket \rightarrow d^n \\ \Gamma_0^n \vdash \Phi_\rho^n &: \Theta_\rho^n \llbracket v_0^n \rrbracket \rightarrow \Theta_{n-1,0}^n \\ (\Gamma_0^n, w : I_0^{n-1}) \vdash \Xi_{n-1,0}^n &: \Theta_{n-1,0}^n \llbracket v_0^n \rrbracket *_{n-1} \Theta_{n-1,0}^n \llbracket w \rrbracket \\ &\rightarrow \Theta_{n-1,0}^n \llbracket v_0^n *_{n-1} w \rrbracket \end{aligned}$$

We then construct the cell $H_{n-1,0}^n$, as the following 4-ary composite, using the above ingredients, as described in Fig. 5:

$$\begin{aligned} &a *_{n-1} b \\ &\quad \downarrow (\rho^n)^{-1} \llbracket a \rrbracket *_{n-1} (\lambda^n)^{-1} \llbracket b \rrbracket \\ &\Theta_\rho^n \llbracket a *_{\text{id}_x^n} \rrbracket *_{n-1} \Theta_\lambda^n \llbracket \text{id}_x^n *_{\text{id}_x^n} b \rrbracket \\ &\quad \downarrow \Phi_\rho^n \llbracket a *_{\text{id}_x^n} \rrbracket *_{n-1} \Phi_\lambda^n \llbracket \text{id}_x^n *_{\text{id}_x^n} b \rrbracket \\ &\Theta_{n-1,0}^n \llbracket a *_{\text{id}_x^n} \rrbracket *_{n-1} \Theta_{n-1,0}^n \llbracket \text{id}_x^n *_{\text{id}_x^n} b \rrbracket \\ &\quad \downarrow \Xi_{n-1,0}^n \llbracket a *_{\text{id}_x^n}, \text{id}_x^n *_{\text{id}_x^n} b \rrbracket \\ &\Theta_{n-1,0}^n \llbracket (a *_{\text{id}_x^n}) *_{n-1} (\text{id}_x^n *_{\text{id}_x^n} b) \rrbracket \\ &\quad \downarrow (\Theta_{n-1,0}^n \uparrow v_0^n) \llbracket \zeta^n \rrbracket \\ &\Theta_{n-1,0}^n \llbracket a *_{\text{id}_x^n} b \rrbracket \end{aligned}$$

Here ζ^n is the interchanger defined as the unique coherence of the following type, explicitly constructed in Appendix E.2:

$$(a *_{\text{id}_x^n}) *_{n-1} (\text{id}_x^n *_{\text{id}_x^n} b) \rightarrow a *_{\text{id}_x^n} b$$

Similarly, for $H_{0,n-1}^n$ recall the cells:

$$\begin{aligned} \mathbb{D}^n \vdash \tilde{\rho}^n &: \Theta_{\tilde{\rho}}^n \llbracket d^n \rrbracket \rightarrow d^n *_{\text{id}_{d_+^n}} \\ \mathbb{D}^n \vdash \tilde{\lambda}^n &: \Theta_{\tilde{\lambda}}^n \llbracket d^n \rrbracket \rightarrow \text{id}_{d_+^n} *_{\text{id}_{d_+^n}} d^n \\ \Gamma_0^n \vdash \Phi_{\tilde{\rho}}^n &: \Theta_{\tilde{\rho}}^n \llbracket v_{n-1}^n \rrbracket \rightarrow \Theta_{0,n-1}^n \\ (\Gamma_{n-1}^n, w : I_{n-1}^{n-1}) \vdash \Xi_{0,n-1}^n &: \\ &\Theta_{0,n-1}^n \llbracket v_{n-1}^n \rrbracket *_{n-1} \Theta_{0,n-1}^n \llbracket w \rrbracket \rightarrow \Theta_{n-1,0}^n \llbracket v_{n-1}^n *_{n-1} w \rrbracket \end{aligned}$$

We then define $H_{0,n-1}^n$ as the following composite:

$$\begin{aligned}
& a *_0 b \\
& \quad \downarrow (\zeta^n)^{-1} \\
& (a *_0 \text{id}_x^n) *_n (\text{id}_x^n *_0 b) \\
& \quad \downarrow (\bar{\rho}^n)^{-1} \llbracket a \rrbracket *_n (\bar{\lambda}^n)^{-1} \llbracket b \rrbracket \\
& \Theta_\lambda^n \llbracket a \rrbracket *_n (\Theta_\lambda^n)^{\text{op}} \llbracket b \rrbracket \\
& \quad \downarrow \Phi_{\bar{\rho}}^n \llbracket a \rrbracket *_n \Phi_{\bar{\lambda}}^n \llbracket b \rrbracket \\
& \Theta_{0,n-1}^n \llbracket a \rrbracket *_n \Theta_{0,n-1}^n \llbracket b \rrbracket \\
& \quad \downarrow \Xi_{0,n-1}^n \llbracket a, b \rrbracket \\
& \Theta_{0,n-1}^n \llbracket a *_n b \rrbracket
\end{aligned}$$

□

Lemma 4.2. *Assuming that a cell $H_{k,l}^n$ satisfying the judgement (2) is defined, we can define a cell $H_{k+1,l+1}^{n+1}$ such that:*

$$\mathbb{E}^{n+1} \vdash H_{k+1,l+1}^{n+1} : E_{k+1,m+1}^{n+1}$$

Proof. The cell $\Sigma H_{k,l}^n \llbracket a, b \rrbracket$ is of type:

$$a *_k b \mapsto (\Sigma \Theta_{k,l}^n) \llbracket a *_k b \rrbracket$$

To obtain a cell of the desired type, we use the morphism of filtrations:

$$\begin{aligned}
\psi_\Sigma : \Gamma_{k+1,l+1}^{n+1} &\rightarrow \Sigma \Gamma_{k,l}^n \\
\Sigma v_l^i \llbracket \psi_\Sigma^{i+1} \rrbracket &= v_{l+1}^{i+1}
\end{aligned}$$

We then define repadding data $\mathbf{r}_{k,l}^n = (f_{k,l}^i, g_{k,l}^i)_{i=m+1}^n$ from $\Sigma \mathbf{u}_{k,l}^n \llbracket \psi_\Sigma \rrbracket$ to $\mathbf{u}_{k+1,l+1}^{n+1}$, whose associated repadding is denoted $\Phi_{k,l}^i$. We define this repadding as follows, denoting $j = i + 1$:

$$\begin{aligned}
f_{k,l}^i &:= \text{coh}(\mathbb{P} : \Sigma p_{k,l}^{i-1} \llbracket \psi_\Sigma^i \rrbracket *_j \Phi_{k,l}^i \llbracket \text{in}^- \circ \sigma^j \rrbracket \rightarrow p_{k+1,l+1}^i) \llbracket x \rrbracket \\
g_{k,l}^i &:= \text{coh}(\mathbb{P} : \Sigma q_{k,l}^{i-1} \llbracket \psi_\Sigma^i \rrbracket \rightarrow \Phi_{k,l}^i \llbracket \text{in}^+ \circ \sigma^j \rrbracket *_j q_{k+1,l+1}^i) \llbracket x \rrbracket
\end{aligned}$$

The associated repadding then has type:

$$\Gamma_{l+1}^{n+1} \vdash \Phi_{k,l}^n : (\Sigma \Theta_{k,l}^n) \llbracket v_{l+1}^{n+1} \rrbracket \rightarrow \Theta_{k+1,l+1}^{n+1}$$

We thus define the cell as follows:

$$H_{k+1,l+1}^{n+1} := \Sigma H_{k,l}^n *_n \Phi_{k,l}^n \quad \square$$

Lemma 4.3. *Assuming that a cell $H_{k,l}^n$ satisfying the judgement (2) is defined, we can define a cell $H_{k,l}^{n+1}$ such that:*

$$\mathbb{E}^{n+1} \vdash H_{k,l}^{n+1} : E_{k,l}^{n+1}$$

Proof. We construct the cell as the following composite:

$$\begin{array}{c}
a *_k b \\
\downarrow (\text{unitor}) \\
(a *_k b) *_n \text{id}(\text{id}_x^n *_k \text{id}_x^n) \\
\downarrow (a *_k b) *_n \xi \\
(a *_k b) *_n (\mathbb{H}_{k,l}^n \llbracket \text{id}_x^n, \text{id}_x^n \rrbracket *_n q_{k,l}^n) \\
\downarrow (\text{associator}) \\
((a *_k b) *_n \mathbb{H}_{k,l}^n \llbracket \text{id}_x^n, \text{id}_x^n \rrbracket) *_n q_{k,l}^n \\
\downarrow (\text{naturality}) \\
(\mathbb{H}_{k,l}^n \llbracket \text{id}_x^n, \text{id}_x^n \rrbracket *_n (\Theta_{k,l}^n \uparrow v_l^n) \llbracket a *_l b \rrbracket) *_n q_{k,l}^n \\
\downarrow (\text{associator}) \\
\mathbb{H}_{k,l}^n \llbracket \text{id}_x^n, \text{id}_x^n \rrbracket *_n (\Theta_{k,l}^n \uparrow v_l^n) \llbracket a *_l b \rrbracket *_n q_{k,l}^n \\
\downarrow \xi' *_n q_{k,l}^n \\
p_{k,l}^n *_n (\Theta_{k,l}^n \uparrow v_l^n) \llbracket a *_l b \rrbracket *_n q_{k,l}^n
\end{array}$$

The step labelled “naturality” is an application of the inverse of naturality of the cell $\mathbb{H}_{k,l}^n$, and ξ and ξ' are the unique coherences of the required type in the context \mathbb{P} . \square

Theorem 4.4. *For every $0 \leq k, l < n$ with $k \neq l$, we can construct a cell $\mathbb{H}_{k,l}^n$ such that:*

$$\mathbb{E}^n \vdash \mathbb{H}_{k,l}^n : E_{k,l}^n$$

*This witnesses that $a *_k b$ is congruent to $a *_l b$.*

Proof. This is obtained by Lemmas 4.1, 4.2 and 4.3 \square

Corollary 4.5. *Given $0 \leq k, l < n$ with $k \neq l$, we construct cells $\text{EH}_{k,l}^n$ such that the following judgements are derivable:*

$$\mathbb{E}^n \vdash \text{EH}_{k,l}^n : a *_k b \rightarrow b *_k a$$

Proof. We make the following definition:

$$\text{EH}_{k,l}^n := \mathbb{H}_{k,l}^n \llbracket a, b \rrbracket *_n ((\mathbb{H}_{k,l}^n)^{\text{op}\{l+1\}} \llbracket b, a \rrbracket)^{-1}$$

The judgement follows from Theorem 4.4 and Lemma 3.3. \square

We can also extend our construction of $\mathbb{H}_{k,l}^n$ to include the case where a padding $\Theta_{p,-}^n$ appears in both the source and the target, as in the vertical morphisms appearing in the Eckmann-Hilton sphere in Fig. 3.

Corollary 4.6. *For $n \in \mathbb{N}$ and $p, k, l \leq n$ with $k \neq l$, there exist terms:*

$$\mathbb{E}^n \vdash \mathbb{H}_{p,k,l}^n : \Theta_{p,k}^n \llbracket a *_k b \rrbracket \rightarrow \Theta_{p,l}^n \llbracket a *_l b \rrbracket$$

Proof. If $p = k$, we define $H_{k,k,l}^n := H_{k,l}^n$, and if $p = l$, we define $H_{l,k,l}^n := (H_{l,k}^n)^{-1}$. Suppose that p, k, l are pairwise disjoint, then by Prop. 3.15, we get terms $\mu_{p,k,l}^n := \mu_{\mathbf{u}_{p,k}^n, \mathbf{u}_{k,l}^n}^n \llbracket a * l b \rrbracket$. In context Γ_l^n , the term $\mu_{p,k,l}^n$ has type:

$$\Theta_{p,k}^n \llbracket \Theta_{k,l}^n \llbracket a * l b \rrbracket \rrbracket \rightarrow \Theta_{\mathbf{u}_{p,k}^n, \square \mathbf{u}_{k,l}^n}^n \llbracket a * l b \rrbracket$$

We then define repadding data in the point context $\mathbb{P} = (x : \star)$:

$$\begin{aligned} f_{p,k,l}^i &:= \text{coh}(\mathbb{P} : p_{p,k}^i \boxplus p_{k,l}^i * \Phi_{\mathbf{m}_{p,k,l}^i}^i \llbracket \text{in}^- \circ \sigma^{i+1} \rrbracket \rightarrow p_{p,l}^i)(x) \\ g_{p,k,l}^i &:= \text{coh}(\mathbb{P} : q_{k,l}^i \boxplus q_{p,k}^i \rightarrow \Phi_{\mathbf{m}_{p,k,l}^i}^i \llbracket \text{in}^+ \circ \sigma^{i+1} \rrbracket * q_{p,m}^i)(x) \end{aligned}$$

We denote $\Phi_{p,k,l}^n$ the repadding associated with this repadding data. We then have that in context Γ_l^n , we have, the term $\Phi_{p,k,l}^n$ has type:

$$\Theta_{\mathbf{u}_{p,k}^n, \square \mathbf{u}_{k,l}^n}^n \llbracket a * l b \rrbracket \rightarrow \Theta_{p,l}^n \llbracket a * l b \rrbracket$$

This lets us define the term $H_{p,k,l}^n$ as follows:

$$(\Theta_{p,k}^n \uparrow v_k^n) \llbracket H_{k,l}^n \rrbracket * \mu_{p,k,l}^n \llbracket a * l b \rrbracket * \Phi_{p,k,l}^n \llbracket a * l b \rrbracket \quad \square$$

5 Implementation

The type theory CaTT is implemented as a proof assistant, which is included in the supplementary material. The proof assistant reads `.catt` files, and typechecks the terms defined therein.

The provided version has our constructions of the cells $H_{k,l}^n$ and $\text{EH}_{k,l}^n$ implemented as new built-in operations, accessible under the names `H` and `EH`. When invoked with suitable arguments, these will trigger our construction to be executed within the proof assistant. As an example, the following commands typecheck and print the terms $H_{2,0}^3$ and $\text{EH}_{2,0}^3$:

```
check H(3, 2, 0)
check EH(3, 2, 0)
```

Using this automation, we can easily compare the size of the terms generated as n, k , and l vary. To assess the complexity of the terms we produce, we use an output method where all the subterms are recursively defined through “let-in” definitions. If a subterm appears multiple times, it is only defined once, and the corresponding name is reused. This allows us to factor out the high degree of repetition in the terms we produce, thus giving a reasonable lower-bound of the work a user would have to do to define those terms manually.

We have used this method to generate a range of pre-computed output artifacts, which are included in the supplementary material in the directory `results`. As $n - \max\{k, l\}$ increases, the size of the output artifact grows rapidly, and we find that terms with $n - \max\{k, l\} > 4$ are typically too large to be computed and type-checked on our available resources, due to the memory overhead required by the type-checker. Performance analysis indicates that it is the

naturality step that dominates the complexity of the proof terms in the limit. In Figure 4, we list the sizes of a variety of artifacts that we have constructed.

Types in Martin-Löf Type Theory have the structure of an ω -groupoid [2, 13, 26], and this has been exploited by Benjamin to implement a pipeline that can convert CaTT terms to elements of identity types in homotopy type theory, within the prover Rocq [7]. We have run this on a selection of our generated terms, and in each case Rocq has successfully validated the resulting structures. It may be interesting to explore opportunities to integrate such Rocq outputs as part of larger proof terms in homotopy type theory.

References

- [1] S. Abramsky and B. Coecke. ‘A categorical semantics of quantum protocols’. In: *Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science, 2004*. 2004, pp. 415–425. DOI: 10.1109/LICS.2004.1319636.
- [2] Thorsten Altenkirch and Ondrej Rypacek. ‘A Syntactical Approach to Weak ω -Groupoids’. In: *21st Annual Conference of the EACSL (CSL 2012)*. 2012. DOI: 10.4230/LIPICS.CSL.2012.16.
- [3] Dimitri Ara. ‘Sur les ∞ -groupoïdes de Grothendieck et une variante ∞ -catégorique’. Thèse de doctorat. Université Paris Diderot (Paris 7), 2010.
- [4] Dimitri Ara et al. *Polygraphs: From Rewriting to Higher Categories*. 2023. arXiv: 2312.00429.
- [5] Michael A. Batanin. ‘Monoidal Globular Categories As a Natural Environment for the Theory of Weak n -Categories’. In: *Advances in Mathematics* 136.1 (1998), pp. 39–103. DOI: 10.1006/aima.1998.1724.
- [6] Thibaut Benjamin. ‘A type theoretic approach to weak ω -categories and related higher structures’. Thèse de doctorat. Institut Polytechnique de Paris, 2020.
- [7] Thibaut Benjamin. ‘Generating Higher Identity Proofs in Homotopy Type Theory’. 2024. arXiv: 2412.01667.
- [8] Thibaut Benjamin and Ioannis Markakis. *Hom ω -Categories of a Computad Are Free*. 2024. arXiv: 2402.01611.
- [9] Thibaut Benjamin and Ioannis Markakis. ‘Invertible cells in ω -categories’. 2024. arXiv: 2406.12127.
- [10] Thibaut Benjamin, Ioannis Markakis and Chiara Sarti. ‘CaTT contexts are finite computads’. In: *Electronic Notes in Theoretical Informatics and Computer Science*. Vol. 4 - Proceedings of MFPS XL. 2024, 5. DOI: 10.46298/entics.14675.
- [11] Thibaut Benjamin, Samuel Mimram and Eric Finster. ‘Globular Weak ω -Categories as Models of a Type Theory’. In: *Higher Structures* 8.2 (2024), pp. 1–69. DOI: 10.21136/HS.2024.07.
- [12] Thibaut Benjamin et al. *Naturality for higher-dimensional path types*. 2025. arXiv: 2501.11620.

- [13] Benno van den Berg and Richard Garner. ‘Types are weak ω -groupoids’. In: *Proceedings of the London Mathematical Society* 102.2 (2011), pp. 370–394. DOI: 10.1112/plms/pdq026.
- [14] John Bourke. ‘Iterated algebraic injectivity and the faithfulness conjecture’. In: *Higher Structures* 4.2 (2020), pp. 183–210. DOI: 10.21136/HS.2020.13.
- [15] Eugenia Cheng. ‘An ω -category with all Duals is an ω -groupoid’. In: *Applied Categorical Structures* 15.4 (2007), pp. 439–453. DOI: 10.1007/s10485-007-9081-8.
- [16] Eugenia Cheng and Alex Corner. *A Higher-Dimensional Eckmann–Hilton Argument*. 2024. URL: <https://alex-corner.github.io/slides/corner-ct24.pdf>.
- [17] Christopher J. Dean et al. ‘Computads for weak ω -categories as an inductive type’. In: *Advances in Mathematics* 450 (2024), p. 109739. DOI: 10.1016/j.aim.2024.109739.
- [18] Peter Dybjer. ‘Internal type theory’. In: *Types for Proofs and Programs*. Ed. by Stefano Berardi and Mario Coppo. Red. by Gerhard Goos, Juris Hartmanis and Jan Leeuwen. Vol. 1158. Berlin, Heidelberg: Springer Berlin Heidelberg, 1996, pp. 120–134. DOI: 10.1007/3-540-61780-9_66.
- [19] Beno Eckmann and Peter J. Hilton. ‘Structure maps in group theory’. In: *Fundamenta Mathematicae* 50 (1961), pp. 207–221. DOI: 10.4064/fm-50-2-207-221.
- [20] Eric Finster and Samuel Mimram. ‘A type-theoretical definition of weak ω -categories’. In: *Proceedings of the 32nd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS 2017)*. ACM, 2017, pp. 1–12. DOI: 10.5555/3329995.3330059.
- [21] Eric Finster et al. ‘A Type Theory for Strictly Unital ∞ -Categories’. In: *Proceedings of the 37th Annual ACM / IEEE Symposium on Logic in Computer Science (LICS 2022)*. 2022, pp. 1–12. DOI: 10.1145/3531130.3533363.
- [22] Soichiro Fujii, Keisuke Hoshino and Yuki Maehara. *ω -weak equivalences between weak ω -categories*. 2024. arXiv: 2406.13240.
- [23] Allen Hatcher. *Algebraic topology*. University Press, 2002.
- [24] Martin Hofmann and Thomas Streicher. ‘The groupoid interpretation of type theory’. In: *Twenty Five Years of Constructive Type Theory*. Oxford University Press, 1998. DOI: 10.1093/oso/9780198501275.003.0008.
- [25] Tom Leinster. ‘Operads in Higher-Dimensional Category Theory’. University of Cambridge, 2000. 127 pp. arXiv: math/0011106.
- [26] Peter LeFanu Lumsdaine. ‘Weak ω -Categories from Intensional Type Theory’. In: *Typed Lambda Calculi and Applications*. Vol. 5608. 2009, pp. 172–187. DOI: 10.1007/978-3-642-02273-9_14.
- [27] Mihaly Makkai and Marek Zawadowski. ‘Natural Associativity and Commutativity’. In: *Rice Institute Pamphlet - Rice University Studies* 49.4 (1963).
- [28] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study, 2013.

- [29] Christopher J. Schommer-Pries. ‘The Classification of Two-Dimensional Extended Topological Field Theories’. PhD Thesis. Max Planck Institute for Mathematics, Bonn, 2011.

A Interactions Between Meta-Operations

We record some lemmas about the various interactions between suspensions, opposites, functoriality, and substitutions.

Lemma A.1. *Let $\sigma : \Delta \rightarrow \Gamma$ be a substitution. Then:*

- *If $\Gamma \vdash t : A$, $\Sigma(t[\sigma]) = (\Sigma t)[\Sigma\sigma]$*
- *If $\Gamma \vdash A$, $\Sigma(A[\sigma]) = (\Sigma A)[\Sigma\sigma]$*

Moreover, if $\Gamma \vdash_{\text{ps}}$, then $\Sigma\partial^\pm\Gamma = \partial^\pm\Sigma\Gamma$, and $\Sigma\text{id}_\Gamma = \text{id}_{\Sigma\Gamma}$.

Proof. This was proved by Benjamin [6, Lemma 71]. □

Lemma A.2. *For any family of terms t_0, \dots, t_n in a context Γ such that $t_0 *_k \dots *_k t_n$ is well defined in Γ , the following equality holds:*

$$\Sigma(t_0 *_k \dots *_k t_n) = (\Sigma t_0) *_k \dots *_k (\Sigma t_n)$$

For any term $\Gamma \vdash t : A$, the following equality holds:

$$\Sigma(\text{id}_t^n) = \text{id}_{\Sigma t}^n$$

Proof. For the first claim, we prove the more general statement that for any pasting context Γ , we have $\Sigma \text{comp}_\Gamma = \text{comp}_{\Sigma\Gamma}$. We prove this by induction on the pasting context Γ . If $\Gamma = \mathbb{D}^n$ is a disc, then we note that up to the α -conversion renaming d^n into d^{n+1} , we have $\Sigma\mathbb{D}^n = \mathbb{D}^{n+1}$. Then, up to the same α -conversion, we have: $\Sigma \text{comp}_{\mathbb{D}^n} = d^n = \text{comp}_{\mathbb{D}^{n+1}}$. When Γ is not a disc, we have by induction and Lemma A.1.

$$\begin{aligned} \Sigma \text{comp}_\Gamma &= \text{coh}(\Sigma\Gamma, \Sigma \text{comp}_{\partial-\Gamma} \rightarrow \text{comp}_{\partial+\Gamma})[\Sigma \text{id}_\Gamma] \\ &= \text{coh}(\Sigma\Gamma, \text{comp}_{\partial-\Gamma} \rightarrow \text{comp}_{\partial+\Gamma})[\text{id}_{\Sigma\Gamma}] \\ &= \text{comp}_{\Sigma\Gamma} \end{aligned}$$

Here we use the fact the ∂^\pm reduces the dimension of pasting contexts and that the unique pasting context of dimension 0 is a disc for this induction to be well-founded.

For the second statement, we proceed by induction on n . By definition, $\Sigma(\text{id}_t^0) = \Sigma t = \text{id}_{\Sigma t}^0$. For $n > 0$, we have:

$$\begin{aligned} \Sigma(\text{id}_t^n) &= \text{coh}(\Sigma\mathbb{D}^n : \Sigma d^n \rightarrow \Sigma d^n)[\Sigma \text{id}_t^{n-1}] \\ &= \text{coh}(\mathbb{D}^{n+1} : d^{n+1} \rightarrow d^{n+1})[\Sigma \text{id}_t^{n-1}] \\ &= \text{coh}(\mathbb{D}^{n+1} : d^{n+1} \rightarrow d^{n+1})[\text{id}_{\Sigma t}^{n-1}] \\ &= \text{id}_{\Sigma t}^n \end{aligned} \quad \square$$

Lemma A.3. *Let $M \subseteq \mathbb{N}_{>0}$ and $\sigma : \Delta \rightarrow \Gamma$. For any term $\Gamma \vdash t : A$, we have:*

$$(t[\sigma])^{\text{op } M} = t^{\text{op } M} [\sigma^{\text{op } M}]$$

Similarly, for any substitution $\tau : \Gamma \rightarrow \Theta$, we have:

$$(\tau \circ \sigma)^{\text{op } M} = \tau^{\text{op } M} \circ \sigma^{\text{op } M}$$

Proof. This is functoriality of the opposites construction, proved by Benjamin and Markakis [8, §5.2], together with the fact that well-typed terms $\Gamma \vdash t : A$ of dimension n are in bijection with substitutions $\Gamma \vdash \sigma : \mathbb{D}^n$. \square

Lemma A.4. *Let Γ be a context and $M \subseteq \mathbb{N}_{>0}$, for any family t_0, \dots, t_n of terms in Γ such that $t_0 *_k \dots *_k t_n$ is well defined, we have:*

$$(t_0 *_k \dots *_k t_n)^{\text{op } M} = \begin{cases} (t_0)^{\text{op } M} *_k \dots *_k (t_n)^{\text{op } M} & k+1 \notin M \\ (t_n)^{\text{op } M} *_k \dots *_k (t_0)^{\text{op } M} & k+1 \in M \end{cases}$$

For any term t in Γ , we have

$$(\text{id}_t^m)^{\text{op } M} = \text{id}_{t^{\text{op } M}}^m$$

Proof. We first prove the first claim, by proving a more general statement, that is, for any pasting context Γ , we have $(\text{comp}_\Gamma)^{\text{op } M} = \text{comp}_{\Gamma'} \llbracket \gamma^{-1} \rrbracket$, where Γ' is the unique pasting context isomorphic to $\Gamma^{\text{op } M}$ and γ is the isomorphism. We prove this by induction on Γ . First when $\Gamma = \mathbb{D}^n$ is a disc, we have $(\mathbb{D}^n)' = \mathbb{D}^n$, with the isomorphism γ swapping d_-^k and d_+^k for every $k \in M$ such that $k < n$, and acting as the identity on all other variables. Thus, we have

$$(\text{comp}_\Gamma)^{\text{op } M} = d^n = d^n \llbracket \gamma^{-1} \rrbracket = \text{comp}_{\Gamma'} \llbracket \gamma^{-1} \rrbracket$$

If Γ is not a disc, we distinguish two cases. If $\dim(\Gamma) \notin M$, then we have the equality $\partial^\pm(\Gamma') = (\partial^\pm \Gamma)'$ [8, Lemma 16], and thus:

$$\begin{aligned} & (\text{comp}_\Gamma)^{\text{op } M} \\ &= \text{coh}(\Gamma' : (\text{comp}_{\partial-\Gamma})^{\text{op } M} \llbracket \gamma \rrbracket \rightarrow (\text{comp}_{\partial+\Gamma})^{\text{op } M} \llbracket \gamma \rrbracket) \llbracket \gamma^{-1} \rrbracket \\ &= \text{coh}(\Gamma' : \text{comp}_{(\partial-\Gamma)'} \rightarrow \text{comp}_{(\partial+\Gamma)'}) \llbracket \gamma^{-1} \rrbracket \\ &= \text{coh}(\Gamma' : \text{comp}_{\partial-(\Gamma')} \rightarrow \text{comp}_{\partial+(\Gamma')}) \llbracket \gamma^{-1} \rrbracket \\ &= \text{comp}_{\Gamma'} \llbracket \gamma^{-1} \rrbracket \end{aligned}$$

On the other hand, if $\dim(\Gamma) \in M$ then we have the equality $\partial^\pm(\Gamma') = (\partial^\mp \Gamma)'$ [8, Lemma 16], and thus:

$$\begin{aligned} & (\text{comp}_\Gamma)^{\text{op } M} \\ &= \text{coh}(\Gamma' : (\text{comp}_{\partial+\Gamma})^{\text{op } M} \llbracket \gamma \rrbracket \rightarrow (\text{comp}_{\partial-\Gamma})^{\text{op } M} \llbracket \gamma \rrbracket) \llbracket \gamma^{-1} \rrbracket \\ &= \text{coh}(\Gamma' : \text{comp}_{(\partial+\Gamma)'} \rightarrow \text{comp}_{(\partial-\Gamma)'}) \llbracket \gamma^{-1} \rrbracket \\ &= \text{coh}(\Gamma' : \text{comp}_{\partial+(\Gamma')} \rightarrow \text{comp}_{\partial-(\Gamma')}) \llbracket \gamma^{-1} \rrbracket \\ &= \text{comp}_{\Gamma'} \llbracket \gamma^{-1} \rrbracket \end{aligned}$$

As before, we use the fact the ∂^\pm reduces the dimension of pasting contexts and that the unique pasting context of dimension 0 is a disc for this induction to be well-founded.

For the second statement, we proceed by induction on m . When $m = 0$, we have $(\text{id}_t^m)^{\text{op } M} = t^{\text{op } M} = \text{id}_{t^{\text{op } M}}^0$ as required. When $m > 0$, we have:

$$\begin{aligned} (\text{id}_t^m)^{\text{op } M} &= \text{coh}((\mathbb{D}^m)' : d^m \llbracket \gamma \rrbracket \rightarrow d^m \llbracket \gamma \rrbracket) [(\text{id}_t^{m-1})^{\text{op } N}] \\ &= \text{coh}(\mathbb{D}^m : d^m \rightarrow d^m) [(\text{id}_t^{m-1})^{\text{op } M}] \\ &= \text{coh}(\mathbb{D}^m : d^m \rightarrow d^m) [\text{id}_{t^{\text{op } M}}^{m-1}] \\ &= \text{id}_{t^{\text{op } M}}^m \quad \square \end{aligned}$$

Lemma A.5. *Let Γ be a context and $X \subseteq \text{Var}(\Delta)$ a set of maximal-dimension variables. Then for any term $\Gamma \vdash t : A$ such that $\text{supp}(t) \cap X = \emptyset$, we have $t \llbracket \text{in}^\pm \rrbracket = t$. Moreover, for any substitution $\Gamma \vdash \sigma : \Delta$ such that $\text{supp}(\sigma) \cap X = \emptyset$, we have $\sigma \circ \text{in}^\pm = \sigma$.*

Proof. We prove these two results by mutual induction. For a term $t = x$ which is a variable, by hypothesis, $x \notin X$, so $x \llbracket \text{in}^\pm \rrbracket = x$. For the term $\text{coh}(\Theta : B) \llbracket \tau \rrbracket$ we have that $\text{supp}(t) = \text{supp}(\tau)$, and we see thus by induction that:

$$t \llbracket \text{in}^\pm \rrbracket = \text{coh}(\Theta : B) \llbracket \tau \circ \text{in}^\pm \rrbracket = t$$

For the empty substitution $\langle \rangle$, we have $\langle \rangle \circ \text{in}^\pm = \langle \rangle$. For the substitution $\langle \sigma, x \mapsto t \rangle$, we have $\text{supp}(t) \subseteq \text{supp}(\sigma)$, thus by induction,

$$\langle \sigma, x \mapsto t \rangle \circ \text{in}^\pm = \langle \sigma \circ \text{in}^\pm, x \mapsto t \llbracket \text{in}^\pm \rrbracket \rangle = \langle \sigma, t \rangle \quad \square$$

Lemma A.6. *Let Δ be a context. Then for any term $\Delta \vdash t : A$, we have that $\text{supp}(A) \subseteq \text{supp}(t)$. Moreover, for any term $\Delta \vdash t : A$, and any substitution $\Gamma \vdash \sigma : \Delta$, we have:*

$$\begin{aligned} \text{supp}(t \llbracket \sigma \rrbracket) &= \bigcup_{y \in \text{supp}(t)} \text{supp}(y \llbracket \sigma \rrbracket) \\ \text{supp}(A \llbracket \sigma \rrbracket) &= \bigcup_{y \in \text{supp}(A)} \text{supp}(y \llbracket \sigma \rrbracket) \end{aligned}$$

Proof. This was proved by Dean et al. [17, Lemma 7.3]. □

Lemma A.7. *Let $\Gamma \vdash \sigma : \Delta$ be a substitution. Let $X \subseteq \text{Var}(\Gamma)$ be an up-closed set of variables of depth at most 1 in Γ and σ . Then the inclusions $\Delta \uparrow X_\sigma \vdash \text{in}_\Delta^\pm : \Delta$ and $\Gamma \uparrow X \vdash \text{in}_\Gamma^\pm : \Gamma$ satisfy:*

$$\text{in}_\Delta^\pm \circ (\sigma \uparrow X) = \sigma \circ \text{in}_\Gamma^\pm$$

Proof. We will show that they coincide on every variable x . If $x \notin X_\sigma$, then by definition

$$x \llbracket \text{in}_\Delta^\pm \circ \sigma \uparrow X \rrbracket = x \llbracket \sigma \rrbracket$$

Since $\text{supp}(x \llbracket \sigma \rrbracket) \cap X = \emptyset$, by Lemma A.5

$$x \llbracket \sigma \circ \text{in}_\Gamma^\pm \rrbracket = x \llbracket \sigma \rrbracket$$

proving the equality. If $x \in X_\sigma$, then by definition,

$$x[\text{in}_\Delta^\pm \circ (\sigma \uparrow X)] = x^\pm[\sigma \uparrow X] = x[\sigma \circ \text{in}_\Gamma^\pm] \quad \square$$

Lemma A.8. *Let $\Gamma \vdash \sigma : \Delta$ a substitution between contexts of the same dimension, and $X \in \text{Var}(\Gamma)$ a set of variables of depth 0 with respect to Γ and σ . Then for any term $\Delta \vdash t : A$ such that $\text{depth}_X t[\sigma] = 0$, we have*

$$\begin{aligned} (A \uparrow^t X_\sigma)[\sigma \uparrow X] &= A[\sigma] \uparrow^{t[\sigma]} X \\ (t \uparrow X_\sigma)[\sigma \uparrow X] &= t[\sigma] \uparrow X \end{aligned}$$

Similarly, for any substitution $\Delta \vdash \tau : \Theta$ such that $\text{depth}_X(\tau \circ \sigma) = 0$, we have:

$$(\tau \uparrow X_\sigma) \circ (\sigma \uparrow X) = (\tau \circ \sigma) \uparrow X$$

Proof. The equality of types is a consequence of Lemma A.7, along with the fact that since A has disjoint support from X , we have $A[\sigma \uparrow X] = A[\sigma]$:

$$\begin{aligned} (A \uparrow^t X_\sigma)[\sigma \uparrow X] &= t[\text{in}^- \circ (\sigma \uparrow X)] \rightarrow_{A[\sigma]} t[\text{in}^+ \circ (\sigma \uparrow X)] \\ &= t[\sigma \circ \text{in}^-] \rightarrow_{A[\sigma]} t[\sigma \circ \text{in}^+] \\ &= (A[\sigma] \uparrow^{t[\sigma]} X) \end{aligned}$$

We prove the equalities on terms and substitution by mutual induction. For a term $t = x$ which is a variable, if $x \in X_\sigma$ then by definition:

$$(x \uparrow X_\sigma)[\sigma \uparrow X] = x[\sigma] \uparrow X$$

If $x \notin X_\sigma$ then

$$(x \uparrow X_\sigma)[\sigma \uparrow X] = x[\sigma]$$

whereas and since $x[\sigma] \cap X = \emptyset$, we also have

$$x[\sigma] \uparrow X = x[\sigma]$$

For the term $\text{coh}(\Theta : B)[\tau]$, then if $X_{\tau \circ \sigma} \neq \emptyset$ we have by induction, denoting $u = \text{coh}(\Theta : B)[\text{id}_\Theta]$:

$$\begin{aligned} (t \uparrow X_\sigma)[\sigma \uparrow X] &= \text{coh}(\Theta \uparrow (X_\sigma)_\tau : u[\text{in}^-] \rightarrow u[\text{in}^+])[(\tau \uparrow X_\sigma) \circ (\sigma \uparrow X)] \\ &= \text{coh}(\Theta \uparrow (X_{\tau \circ \sigma}) : u[\text{in}^-] \rightarrow u[\text{in}^+])[(\tau \circ \sigma) \uparrow X] \\ &= t[\sigma] \uparrow X \end{aligned}$$

If $X_{\tau \circ \sigma} = \emptyset$, by Lemma A.6 we have $\text{supp}(t[\sigma]) \cap X = \emptyset$. Then by induction together with Lemmas A.5 and A.7, we have:

$$\begin{aligned} (t \uparrow X_\sigma)[\sigma \uparrow X] &= t[\sigma \uparrow X] \\ &= t[\text{in}_\Delta^\pm \circ \sigma \uparrow X] \\ &= t[\sigma \circ \text{in}_\Gamma^\pm] \uparrow X \\ &= t[\sigma] \end{aligned}$$

For the empty substitution $\langle \rangle$, we have:

$$\langle \rangle \uparrow X_\sigma \circ (\sigma \uparrow X) = \langle \rangle = (\langle \rangle \circ \sigma) \uparrow X$$

For substitutions of the form $\langle \tau, x \mapsto t \rangle$, if $x \in X_{\tau \circ \sigma}$, we have, by induction and Lemma A.7:

$$\begin{aligned} & \langle \tau, x \mapsto t \rangle \uparrow X_\sigma \circ (\sigma \uparrow X) \\ &= \left\langle \begin{array}{l} (\tau \uparrow X_\sigma) \circ (\sigma \uparrow X), x^\pm \mapsto t[\text{in}^\pm \circ (\sigma \uparrow X)], \\ \vec{x} \mapsto (t \uparrow X_\sigma)[\sigma \uparrow X] \end{array} \right\rangle \\ &= \left\langle \begin{array}{l} (\tau \circ \sigma) \uparrow X, x^\pm \mapsto t[\text{in}^\pm \circ (\sigma \uparrow X)], \\ \vec{x} \mapsto (t \uparrow X_\sigma)[\sigma \uparrow X] \end{array} \right\rangle \\ &= \langle (\tau \circ \sigma) \uparrow X, x^\pm \mapsto t[\sigma \circ \text{in}^\pm], \vec{x} \mapsto (t \uparrow X_\sigma)[\sigma \uparrow X] \rangle \\ &= \langle (\tau \circ \sigma) \uparrow X, x^\pm \mapsto t[\sigma \circ \text{in}^\pm], \vec{x} \mapsto t[\sigma] \uparrow X \rangle \\ &= \langle \langle \tau, x \mapsto t \rangle \circ \sigma \rangle \uparrow X \end{aligned}$$

On the other hand, if $x \notin X_{\tau \circ \sigma}$, we have by induction,

$$\begin{aligned} & \langle \tau, x \mapsto t \rangle \uparrow X_\sigma \circ (\sigma \uparrow X) \\ &= \langle \tau \uparrow X_\sigma \circ (\sigma \uparrow X), x \mapsto t[\sigma \uparrow X] \rangle \\ &= \langle (\tau \circ \sigma) \uparrow X, x \mapsto t[\sigma \uparrow X] \rangle \\ &= \langle \langle \tau, x \mapsto t \rangle \circ \sigma \rangle \uparrow X \end{aligned} \quad \square$$

Lemma A.9. *For every context $\Gamma \vdash$ and every $X \in \text{Up}(\Gamma)$ such that $\text{depth}_X(\Gamma) = 0$:*

$$\begin{aligned} \Sigma(\Gamma \uparrow X) &= (\Sigma\Gamma) \uparrow X \\ \Sigma \text{in}_{\Gamma, X}^\pm &= \text{in}_{\Sigma\Gamma, X}^\pm \end{aligned}$$

Proof. We proceed by structural induction on Γ . For the empty context \emptyset , we have:

$$\Sigma(\emptyset \uparrow \emptyset) = (\Sigma\emptyset) \uparrow \emptyset = (N : \star, S : \star).$$

For the context $(\Gamma, x : A)$, denote $X' = X \setminus \{x\}$. Then, if $x \in X$ we have:

$$\begin{aligned} \Sigma((\Gamma, x : A) \uparrow X) &= \Sigma(\Gamma \uparrow X', x^\pm : A, \vec{x} : x^- \rightarrow_A x^+) \\ &= (\Sigma(\Gamma \uparrow X'), x^\pm : \Sigma A, \vec{x} : x^- \rightarrow_{\Sigma A} x^+) \\ &= ((\Sigma\Gamma) \uparrow X', x^\pm : \Sigma A, \vec{x} : x^- \rightarrow_{\Sigma A} x^+) \\ &= \Sigma(\Gamma, x : A) \uparrow \Sigma X \end{aligned}$$

On the other hand, if $x \notin X$, we have:

$$\begin{aligned} \Sigma((\Gamma, x : A) \uparrow X) &= \Sigma(\Gamma \uparrow X', x : A) \\ &= (\Sigma(\Gamma \uparrow X'), x : \Sigma A) \\ &= ((\Sigma\Gamma) \uparrow X', x : \Sigma A) \\ &= \Sigma(\Gamma, x : A) \uparrow \Sigma X \end{aligned}$$

For the second statement, consider a variable x of Γ . If $x \notin X$, then we have:

$$x[\text{in}_{\Sigma\Gamma, X}^\pm] = x = \Sigma(x[\text{in}_{\Gamma, X}^\pm]).$$

If $x \in X$, then:

$$x[\text{in}_{\Sigma\Gamma, X}^\pm] = x^\pm = x[\Sigma \text{in}_{\Gamma, X}^\pm].$$

Finally, since $N, S \notin X$, we have:

$$\begin{aligned} N[\text{in}_{\Sigma\Gamma, X}^\pm] &= N = N[\Sigma \text{in}_{\Gamma, X}^\pm] \\ S[\text{in}_{\Sigma\Gamma, X}^\pm] &= S = S[\Sigma \text{in}_{\Gamma, X}^\pm] \end{aligned}$$

The two substitutions thus coincide on all variables and therefore are equal. \square

Lemma A.10. *For every context Γ and every $X \in \text{Up}(\Gamma)$ such that $\text{depth}_X(\Gamma) = 0$, the following hold:*

- For any term $\Gamma \vdash t : A$ such that $\text{depth}_X(t) = 0$, we have:

$$\Sigma(t \uparrow X) = (\Sigma t) \uparrow X$$

- For any substitution $\Gamma \vdash \sigma : \Delta$ such that $\text{depth}_X(\sigma) = 0$, we have:

$$\Sigma(\sigma \uparrow X) = (\Sigma\sigma) \uparrow X$$

- For any term $\Gamma \vdash t : A$ such that $\text{depth}_X(t) = 1$, we have:

$$\Sigma(A \uparrow^t X) = (\Sigma A) \uparrow^{\Sigma t} X$$

Proof. We prove the first two statements together by mutual induction. If $t = x$ is a variable in X , then:

$$\Sigma(x \uparrow X) = \vec{x} = (\Sigma x) \uparrow \Sigma X$$

If $t = \text{coh}_{\Delta, B}[\sigma]$ and $\sigma^{-1}X = \emptyset$, then:

$$\Sigma(t \uparrow X) = \Sigma t = (\Sigma t) \uparrow \Sigma X$$

If $\sigma^{-1}X \neq \emptyset$, since $N, S \notin X$, we have that $\sigma^{-1}X = (\Sigma\sigma)^{-1}X$. Denoting $Y = \sigma^{-1}X$ and $u = \text{coh}_{\Delta, B}[\text{id}]$, we have by Lemma A.9:

$$\begin{aligned} \Sigma(t \uparrow X) &= \text{coh}_{\Sigma(\Delta \uparrow Y), (\Sigma u)[\Sigma \text{in}_{\Delta, Y}^-] \rightarrow (\Sigma u)[\Sigma \text{in}_{\Delta, Y}^+]}[\Sigma(\sigma \uparrow X)] \\ &= (\Sigma t) \uparrow \Sigma X \end{aligned}$$

For the second statement, for the empty substitution $\langle \rangle$, we have, since $N, S \notin X$:

$$\Sigma(\langle \rangle \uparrow X) = \langle N \mapsto N, S \mapsto S \rangle = (\Sigma \langle \rangle) \uparrow X$$

For the substitution $\langle \sigma, x \mapsto t \rangle$, if $x \notin X$, we have:

$$\begin{aligned} \Sigma(\langle \sigma, x \mapsto t \rangle \uparrow X) &= \langle \Sigma(\sigma \uparrow X), x \mapsto \Sigma t \rangle \\ &= \langle (\Sigma\sigma) \uparrow X, x \mapsto \Sigma t \rangle \\ &= (\Sigma\langle \sigma, x \mapsto t \rangle) \uparrow X \end{aligned}$$

On the other hand, if $x \in X$, then by the inductive hypothesis, by Lemma A.9, and by the following equation [6, Lemma 71]

$$\Sigma(t \llbracket \text{in}_{\Gamma, X}^{\pm} \rrbracket) = (\Sigma t) \llbracket \Sigma \text{in}_{\Gamma, X}^{\pm} \rrbracket$$

we may compute that:

$$\begin{aligned} \Sigma(\langle \sigma, x \mapsto t \rangle \uparrow X) &= \langle \Sigma(\sigma \uparrow X), x^{\pm} \mapsto \Sigma(t \llbracket \text{in}_{\Gamma, X}^{\pm} \rrbracket), \vec{x} \mapsto \Sigma(t \uparrow X) \rangle \\ &= \langle (\Sigma\sigma) \uparrow X, x^{\pm} \mapsto \Sigma(t \llbracket \text{in}_{\Gamma, X}^{\pm} \rrbracket), \vec{x} \mapsto (\Sigma t) \uparrow X \rangle \\ &= \langle (\Sigma\sigma) \uparrow \Sigma X, x^{\pm} \mapsto (\Sigma t) \llbracket \text{in}_{\Sigma\Gamma, X}^{\pm} \rrbracket, \vec{x} \mapsto (\Sigma t) \uparrow X \rangle \\ &= (\Sigma\langle \sigma, x \mapsto t \rangle) \uparrow X \end{aligned}$$

Finally, for the last statement, write $A = u \rightarrow v$ and $n = \dim A$. If $\text{Var}(v) \cap X = \emptyset$ then the source of $\Sigma(A \uparrow^t X)$ is given by $\Sigma(t \llbracket \text{in}_{\Gamma, X}^{-} \rrbracket)$. On the other hand, the source of $(\Sigma A) \uparrow^{\Sigma t} X$ is $(\Sigma t) \llbracket \text{in}_{\Sigma\Gamma, X}^{-} \rrbracket$. Again by Lemma A.9 and the same equality as above, we may deduce that the two sources agree. If $\text{Var}(v) \cap X = \emptyset$, then the source of $\Sigma(A \uparrow^t X)$ is $\Sigma((t \llbracket \text{in}_{\Gamma, X}^{-} \rrbracket) *_{n} (v \uparrow X))$, while the source of $(\Sigma A) \uparrow^{\Sigma t} X$ is $(\Sigma t) \llbracket \text{in}_{\Sigma\Gamma, X}^{-} \rrbracket *_{n+1} ((\Sigma v) \uparrow X)$. By the first part of the lemma and the same reasoning as in the previous case, we see that the two sources agree. A similar argument shows that the target are also equal, proving that the two types coincide. \square

Lemma A.11. *Let Γ be a n -dimensional context and $X \subseteq \text{Var}(\Gamma)$ a set of variables of dimension n . Then for any $M \subseteq \mathbb{N}_{>0}$, there exists an isomorphism:*

$$\text{op}_{\Gamma, X, M}^{\uparrow} : (\Gamma \uparrow X)^{\text{op } M} \xrightarrow{\sim} (\Gamma^{\text{op } M}) \uparrow X \quad (3)$$

Moreover, the source and target inclusions $\Gamma \uparrow X \vdash \text{in}_{\Gamma}^{\pm} : \Gamma$ and $\Gamma^{\text{op } M} \uparrow X \vdash \text{in}_{\Gamma^{\text{op } M}}^{\pm} : \Gamma^{\text{op } M}$ satisfy

$$(\text{in}_{\Gamma}^{\pm})^{\text{op } M} = \begin{cases} \text{in}_{\Gamma^{\text{op } M}}^{\pm} \circ \text{op}_{\Gamma, X, M}^{\uparrow} & n+1 \notin M \\ \text{in}_{\Gamma^{\text{op } M}}^{\mp} \circ \text{op}_{\Gamma, X, M}^{\uparrow} & n+1 \in M \end{cases} \quad (4)$$

If Γ is a pasting context, denote Γ' the unique pasting context isomorphic to $\Gamma^{\text{op } M}$ and $\Gamma' \vdash \gamma_{\Gamma} : \Gamma^{\text{op } M}$ the associated isomorphism. Similarly, denote by $(\Gamma \uparrow X)'$ the unique pasting context isomorphic to $(\Gamma \uparrow X)^{\text{op } M}$ and denote by

$(\Gamma \uparrow X)' \vdash \gamma_{\Gamma \uparrow X} : \Gamma^{\text{op}M} \uparrow X^{\text{op}M}$ the associated isomorphism, then:

$$(\Gamma \uparrow X)' = \Gamma' \uparrow X \quad (5)$$

$$\gamma_{\Gamma \uparrow X} = \text{op}_{\Gamma, X, M}^{\uparrow} \circ \gamma_{\Gamma \uparrow X} \quad (6)$$

$$\gamma_{\Gamma}^{-1} \circ (\text{in}_{\Gamma^{\text{op}M}}^{\pm}) \circ \gamma_{\Gamma \uparrow X} = \begin{cases} \text{in}_{\Gamma'}^{\pm} & n+1 \notin M \\ \text{in}_{\Gamma'}^{\mp} & n+1 \in M \end{cases} \quad (7)$$

Proof. Before proving the lemma, we note that $\Gamma \uparrow X$ is a pasting context [6, Lemma 87], so $(\Gamma \uparrow X)'$ exists. We first prove the isomorphism (3) by structural induction on Γ . For the empty context \emptyset , we have:

$$(\emptyset \uparrow X)^{\text{op}M} = \emptyset = \emptyset^{\text{op}M} \uparrow X$$

For contexts of the form $(\Gamma, x : A)$, if $x \notin X$, we have:

$$\begin{aligned} ((\Gamma, x : A) \uparrow X)^{\text{op}M} &= ((\Gamma \uparrow X)^{\text{op}M}, x : A^{\text{op}M}) \\ &= (\Gamma^{\text{op}M} \uparrow X, x : A^{\text{op}M}) \\ &= (\Gamma, x : A)^{\text{op}M} \uparrow X \end{aligned}$$

If $x \in X$, and $n+1 \in M$, then we have:

$$\begin{aligned} ((\Gamma, x : A) \uparrow X)^{\text{op}M} \\ = ((\Gamma \uparrow (X \setminus \{x\}))^{\text{op}M}, x^{\pm} : A^{\text{op}M}, \vec{x} : x^+ \rightarrow x^-) \end{aligned}$$

Thus we define the isomorphism:

$$\begin{aligned} \text{op}_{\Gamma, X, M}^{\uparrow} : ((\Gamma, x : A) \uparrow X)^{\text{op}M} &\rightarrow (\Gamma \uparrow, x : A)^{\text{op}M} \\ \langle \text{op}_{\Gamma, X \setminus \{x\}, N}^{\uparrow}, x^{\pm} \mapsto x^{\mp}, \vec{x} \mapsto \vec{x} \rangle \end{aligned}$$

If $x \in X$ and $n+1 \notin M$, then we have:

$$\begin{aligned} ((\Gamma, x : A) \uparrow X)^{\text{op}M} \\ = ((\Gamma \uparrow (X \setminus \{x\}))^{\text{op}M}, x^{\pm} : A^{\text{op}N}, \vec{x} : x^- \rightarrow x^+) \end{aligned}$$

Thus we define the isomorphism:

$$\begin{aligned} \text{op}_{\Gamma, X, M}^{\uparrow} : ((\Gamma, x : A) \uparrow X)^{\text{op}M} &\rightarrow (\Gamma \uparrow, x : A)^{\text{op}M} \\ \langle \text{op}_{\Gamma, X \setminus \{x\}, N}^{\uparrow}, x^{\pm} \mapsto x^{\pm}, \vec{x} \mapsto \vec{x} \rangle \end{aligned}$$

We then prove (4) by showing that the two substitutions coincide on every variable. Let $x \in \text{Var}(\Gamma)$, if $x \notin X$, then we have:

$$x \llbracket (\text{in}_{\Gamma}^{\pm})^{\text{op}M} \rrbracket = x = x \llbracket \text{in}_{\Gamma^{\text{op}M}}^{\pm} \circ \text{op}_{\Gamma, X, M}^{\uparrow} \rrbracket$$

If $x \in X$, and $n+1 \in M$, then we have:

$$x \llbracket (\text{in}_{\Gamma}^{\pm})^{\text{op}M} \rrbracket = x^{\pm} = x \llbracket \text{in}_{\Gamma^{\text{op}N}}^{\pm} \circ \text{op}_{\Gamma, X, M}^{\uparrow} \rrbracket$$

If $x \in X$, and $n + 1 \notin M$, then we have:

$$x \llbracket (\text{in}_\Gamma^\pm)^{\text{op } M} \rrbracket = x^\mp = x \llbracket \text{in}_{\Gamma^{\text{op } N}}^\pm \text{op}_{\Gamma, X, M}^\uparrow \rrbracket$$

Equation (5) follows from (3) by uniqueness, since $\Gamma' \uparrow X$ is a pasting context isomorphic to $\Gamma^{\text{op } M} \uparrow X = (\Gamma \uparrow X)^{\text{op } M}$. Equation (6) is then a consequence of this equality, obtained by noticing that both $\gamma_\Gamma \uparrow X$ and $\text{op}_{\Gamma, X, M}^\uparrow \circ \Gamma \uparrow X$ are isomorphism whose source is the pasting context $(\Gamma \uparrow X)'$. Thus there must be equal since pasting context have no non-trivial automorphisms [20]. Finally, Equation (7) follows from (4) at the level of variables, since by definition the substitutions γ_Γ and $\gamma_{\Gamma \uparrow X}$ act as the identity on every variable. The substitution have the same source and target and coincide on every variable, thus, they are equal. \square

Lemma A.12. *Let Γ be a context, $X \subseteq \text{Var}(\Gamma)$ a set of maximal-dimensional variables of Γ and $M \subseteq \mathbb{N}_{>0}$. For any term $\Gamma \vdash t : A$ such that $\text{depth}_X(t) = 0$, we have that:*

$$\begin{aligned} (A \uparrow^t X)^{\text{op } M} \llbracket \text{op}_{\Gamma, X, M}^\uparrow \rrbracket &= A^{\text{op } M} \uparrow^{t^{\text{op } M}} X \\ (t \uparrow X)^{\text{op } M} \llbracket \text{op}_{\Gamma, X, M}^\uparrow \rrbracket &= t^{\text{op } M} \uparrow X \end{aligned}$$

Moreover for any substitution $\Gamma \vdash \sigma : \Delta$ such that $\text{depth}_X(\sigma) \leq 0$, we have:

$$(\sigma \uparrow X)^{\text{op } M} \circ \text{op}_{\Gamma, X, M}^\uparrow = (\text{op}_{\Delta, X_\sigma, N}^\uparrow)^{-1} \circ \sigma^{\text{op } M} \uparrow X$$

Proof. We prove the statements together by mutual induction. First note that by definition, the substitution $\text{op}_{\Gamma, X, M}^\uparrow$ acts as the identity on every variable that is not in X , and thus for any term whose support does not intersect X , we have we have $t \llbracket \text{op}_{\Gamma, X, M}^\uparrow \rrbracket = t$, and consequently, we have:

$$(t \uparrow X)^{\text{op } M} \llbracket \text{op}_{\Gamma, X, M}^\uparrow \rrbracket = t^{\text{op } M} = t^{\text{op } M} \uparrow X$$

Thus it suffices to prove the result for terms whose support intersect X . For a term $t = x$ which is a variable, necessarily $x \in X$ and thus we have:

$$(x \uparrow X)^{\text{op } M} \llbracket \text{op}_{\Gamma, X, M}^\uparrow \rrbracket = \vec{x} = x \uparrow X$$

For the term $\text{coh}(\Delta : B)[\sigma]$, we denote $u = \text{coh}(\Delta : B)[\text{id}_\Delta]$, and $v = \text{coh}(\Delta' : B^{\text{op } M} \llbracket \gamma_\Delta \rrbracket)[\text{id}_{\Delta'}] = u^{\text{op } N} \llbracket \gamma_\Delta \rrbracket$. We note that by induction and Lemmas A.8 and A.11, we have the equalities:

$$\begin{aligned} \gamma_{\Delta \uparrow X_\sigma}^{-1} \circ (\sigma \uparrow X)^{\text{op } M} \circ (\text{op}_{\Gamma, X, M}^\uparrow) &= (\gamma_\Gamma^{-1} \circ (\sigma \uparrow X))^{\text{op } M} \\ (B \uparrow^u X_\sigma)^{\text{op } M} \llbracket \gamma_{\Delta \uparrow X} \rrbracket &= B^{\text{op } M} \llbracket \gamma_\Delta \rrbracket \uparrow^v X_\sigma \\ (\Delta \uparrow X_\sigma)' &= \Delta' \uparrow X_\sigma \end{aligned}$$

Using this equality together with Lemma (5) shows:

$$\begin{aligned} &(t \uparrow X)^{\text{op } M} \llbracket \text{op}_{\Gamma, X, M}^\uparrow \rrbracket \\ &= \text{coh}(\Delta' \uparrow X_\sigma : (B^{\text{op } M} \llbracket \gamma_\Gamma \rrbracket \uparrow^v X)) \llbracket (\gamma_\Delta^{-1} \circ (\sigma \uparrow X))^{\text{op } M} \rrbracket \\ &= t^{\text{op } M} \uparrow X \end{aligned}$$

Given a term $\Gamma \vdash t : A$, of dimension n , by Lemma A.11, we have, if $n + 1 \in M$:

$$\begin{aligned}
& (A \uparrow^t X)^{\text{op } M} \llbracket \text{op}_{\Gamma, X, M}^\uparrow \rrbracket \\
&= t^{\text{op } M} \llbracket (\text{in}_\Gamma^+)^{\text{op } M} \circ \text{op}_{\Gamma, X, M}^\uparrow \rrbracket \rightarrow t^{\text{op } M} \llbracket (\text{in}_\Gamma^-)^{\text{op } M} \circ \text{op}_{\Gamma, X, M}^\uparrow \rrbracket \\
&= t^{\text{op } M} \llbracket \text{in}_{\Gamma^{\text{op } M}}^+ \rrbracket \rightarrow t^{\text{op } M} \llbracket \text{in}_{\Gamma^{\text{op } M}}^- \rrbracket \\
&= A^{\text{op } M} \uparrow^{t^{\text{op } M}} X
\end{aligned}$$

On the other hand, if $n + 1 \notin M$:

$$\begin{aligned}
& (A \uparrow^t X)^{\text{op } M} \llbracket \text{op}_{\Gamma, X, M}^\uparrow \rrbracket \\
&= t^{\text{op } M} \llbracket (\text{in}_\Gamma^-)^{\text{op } M} \circ \text{op}_{\Gamma, X, M}^\uparrow \rrbracket \rightarrow t^{\text{op } M} \llbracket (\text{in}_\Gamma^+)^{\text{op } M} \circ \text{op}_{\Gamma, X, M}^\uparrow \rrbracket \\
&= t^{\text{op } M} \llbracket \text{in}_{\Gamma^{\text{op } M}}^- \rrbracket \rightarrow t^{\text{op } M} \llbracket \text{in}_{\Gamma^{\text{op } M}}^+ \rrbracket \\
&= A^{\text{op } M} \uparrow^{t^{\text{op } M}} X
\end{aligned}$$

For the empty substitution $\langle \rangle$, we have:

$$(\langle \rangle \uparrow \emptyset)^{\text{op } N} \circ \text{op}_{\emptyset, \emptyset, X}^\uparrow = \langle \rangle = \langle \rangle^{\text{op } N} \uparrow \emptyset$$

For a substitution of the form $\Gamma \vdash \langle \sigma, x \mapsto t \rangle : (\Gamma, x : A)$, if $x \notin X$, we have by induction:

$$\begin{aligned}
\langle \langle \sigma, x \mapsto t \rangle \uparrow X \rangle^{\text{op } M} &= \langle (\sigma \uparrow X)^{\text{op } M}, x \mapsto t^{\text{op } M} \rangle \\
&= \langle \sigma^{\text{op } M} \uparrow X^{\text{op } M}, x \mapsto t^{\text{op } M} \rangle \\
&= \langle \sigma, x \mapsto t \rangle^{\text{op } M} \uparrow X
\end{aligned}$$

If $x \in X$ and $\dim(x) + 1 \notin M$, then we have, by induction and Lemma A.11:

$$\begin{aligned}
& \langle \langle \sigma, x \mapsto t \rangle \uparrow X \rangle^{\text{op } M} \circ \text{op}^\uparrow \\
&= \left\langle \begin{array}{l} (\sigma \uparrow X)^{\text{op } M} \circ \text{op}^\uparrow, x^\pm \mapsto t^{\text{op } M} \llbracket (\text{in}_\Gamma^\pm)^{\text{op } M} \circ \text{op}^\uparrow \rrbracket, \\ \vec{x} \mapsto (t \uparrow X)^{\text{op } M} \llbracket \text{op}^\uparrow \rrbracket \end{array} \right\rangle \\
&= \left\langle \begin{array}{l} \sigma^{\text{op } M} \uparrow X, x^\pm \mapsto t^{\text{op } M} \llbracket \text{in}_\Gamma^\pm \rrbracket, \\ \vec{x} \mapsto t^{\text{op } M} \uparrow X \end{array} \right\rangle \\
&= \langle \sigma, x \mapsto t \rangle^{\text{op } M} \uparrow X
\end{aligned}$$

Similarly, if $\dim(x) + 1 \in M$, then:

$$\begin{aligned}
& \langle \langle \sigma, x \mapsto t \rangle \uparrow X \rangle^{\text{op } M} \circ \text{op}^\uparrow \\
&= \left\langle \begin{array}{l} (\sigma \uparrow X)^{\text{op } M} \circ \text{op}^\uparrow, x^\pm \mapsto t^{\text{op } M} \llbracket (\text{in}_\Gamma^\mp)^{\text{op } M} \circ \text{op}^\uparrow \rrbracket, \\ \vec{x} \mapsto (t \uparrow X)^{\text{op } M} \llbracket \text{op}^\uparrow \rrbracket \end{array} \right\rangle \\
&= \left\langle \begin{array}{l} \sigma^{\text{op } M} \uparrow X, x^\pm \mapsto t^{\text{op } M} \llbracket \text{in}_\Gamma^\pm \rrbracket, \\ \vec{x} \mapsto t^{\text{op } M} \uparrow X \end{array} \right\rangle \\
&= \langle \sigma, x \mapsto t \rangle^{\text{op } M} \uparrow X
\end{aligned}$$

□

B Interactions with Inverses

We record some lemmas about inverses. First, we present the definition.

Definition B.1. We say an coherence term $t = \text{coh}(\Gamma : A)[\sigma]$ is *invertible* if either:

- (a) t is a coherence.
- (b) t is a composite, and σ satisfies the invertible image condition.

Where we say a substitution $\Delta \vdash \sigma : \Gamma$ satisfies the *invertible image condition* if the images of all maximal-dimension variables of Γ under σ are invertible.

The work of Benjamin and Markakis [9] shows that these are exactly the equivalences of CaTT , justifying the definition.

Definition B.2. We define, by mutual induction:

- Let $t = \text{coh}(\Gamma : u \rightarrow v)[\sigma]$ be an invertible term. Let $n = \dim(t)$. We define:

$$t^{-1} := \begin{cases} \text{coh}(\Gamma : v \rightarrow u)[\sigma] & \text{if (a)} \\ \text{coh}(\Gamma' : A^{\text{op}\{n\}} \llbracket \gamma \rrbracket) [\gamma^{-1} \circ \bar{\sigma}] & \text{if (b)} \end{cases}$$

- Let $\Delta \vdash \sigma : \Gamma$ be a substitution satisfying the invertible image condition. Let $n = \dim(\Gamma)$. We define:

$$\Delta \vdash \bar{\sigma} : \Gamma^{\text{op}\{n\}}$$

$$x \llbracket \bar{\sigma} \rrbracket := \begin{cases} (x \llbracket \sigma \rrbracket)^{-1} & \dim(x) = n \\ x \llbracket \sigma \rrbracket & \dim(x) < n \end{cases}$$

Lemma B.3. If $\Delta \vdash t : A$ is invertible, and $\Gamma \vdash \sigma : \Delta$ is a substitution, then $t \llbracket \sigma \rrbracket$ is invertible, and the following also holds:

$$(t \llbracket \sigma \rrbracket)^{-1} = t^{-1} \llbracket \sigma \rrbracket$$

If $\Gamma \vdash \sigma : \Delta$ and $\Delta \vdash \tau : \Theta$ are substitutions, and τ satisfies the invertible image condition, then so does $\tau \circ \sigma$, and the following also holds:

$$\overline{\tau \circ \sigma} = \bar{\tau} \circ \sigma$$

Proof. We prove the two statements by mutual induction. For the term $\text{coh}(\Theta : u \rightarrow v)[\tau]$, if it satisfies (a), then so does $\text{coh}(\Theta : u \rightarrow v)[\tau \circ \sigma] = t \llbracket \sigma \rrbracket$, so it is also invertible, and we have:

$$(t \llbracket \sigma \rrbracket)^{-1} = \text{coh}(\Theta : v \rightarrow u)[\tau \circ \sigma] = t^{-1} \llbracket \sigma \rrbracket$$

If $\text{coh}(\Theta : u \rightarrow v)[\tau]$ satisfies (b), then τ satisfies the invertible image condition, and so by the inductive hypothesis for the second statement, so does $\tau \circ \sigma$, so $\text{coh}(\Theta : u \rightarrow v)[\tau \circ \sigma] = t[\sigma]$ is invertible. Moreover, we have:

$$\begin{aligned} (t[\sigma])^{-1} &= \text{coh}(\Theta' : (u \rightarrow v)^{\text{op}\{n\}}[\gamma])[\gamma^{-1} \circ \overline{\tau \circ \sigma}] \\ &= \text{coh}(\Theta' : (u \rightarrow v)^{\text{op}\{n\}}[\gamma])[\gamma^{-1} \circ \overline{\tau} \circ \sigma] \\ &= t^{-1}[\sigma] \end{aligned}$$

For the empty substitution $\langle \rangle$ which trivially satisfies the invertible image condition, we have:

$$\overline{\langle \rangle \circ \sigma} = \overline{\langle \rangle} = \langle \rangle = \langle \rangle \circ \sigma = \overline{\langle \rangle} \circ \sigma$$

For the substitution $\langle \tau, x \mapsto t \rangle$, then either $\dim(x) = n$ or $\dim(x) < n$. In the first case, we have by induction:

$$\begin{aligned} \overline{\langle \tau, x \mapsto t \rangle \circ \sigma} &= \langle \overline{\tau \circ \sigma}, x \mapsto (t[\sigma])^{-1} \rangle \\ &= \langle \overline{\tau} \circ \sigma, x \mapsto t^{-1}[\sigma] \rangle \\ &= \overline{\langle \tau, x \mapsto t \rangle} \circ \sigma \end{aligned}$$

In the second case, we have by induction:

$$\begin{aligned} \overline{\langle \tau, x \mapsto t \rangle \circ \sigma} &= \langle \overline{\tau \circ \sigma}, x \mapsto t[\sigma] \rangle \\ &= \langle \overline{\tau} \circ \sigma, x \mapsto t[\sigma] \rangle \\ &= \overline{\langle \tau, x \mapsto t \rangle} \circ \sigma \end{aligned} \quad \square$$

C Lemmas about Padding

We recall and prove the following lemmas from Section 3.1.

Lemma 3.6. *Given $\psi : \Delta \rightarrow \Gamma$ a morphism of filtrations, if \mathbf{A} is a type family adapted to Γ , then $\mathbf{A}[\psi]$ is adapted to Δ . If \mathbf{p} is padding data for \mathbf{A} with associated padding $\Theta_{\mathbf{p}}$, then $\mathbf{p}[\psi]$ is padding data for $\mathbf{A}[\psi]$, with associated padding then $\Theta_{\mathbf{p}}[\psi]$.*

Proof. Consider a type family $A^i = s^{i-1} \rightarrow t^{i-1}$ adapted to Γ . Because ψ is a morphism of padding filtrations, $w^m = v^m[\psi^m]$. Since $\Gamma^m \vdash v^m : A^m$ and we have:

$$\Gamma^m \vdash w^m : A^m[\psi^m]$$

In context Γ^{i+1} , the terms s^i and t^i have type $A^i[\sigma^{i+1}]$, so in context Δ^{i+1} , the types $s^i[\psi^{i+1}]$ and $t^i[\psi^{i+1}]$ have type $A^i[\sigma^{i+1} \circ \psi^{i+1}]$. The following equality, proved by (1) then lets us conclude that $\mathbf{A}[\psi]$ is adapted to Δ :

$$A^i[\sigma^{i+1} \circ \psi^{i+1}] = A^i[(\psi^i \uparrow w^i) \circ \tau^{i+1}] = A^i[\psi^i][\tau^{i+1}]$$

Let \mathbf{p} be padding data for \mathbf{A} . In context Γ^{i+1} , the term p^i has type:

$$s^i \rightarrow \Theta_{\mathbf{p}}^i[\text{in}^- \circ \sigma^{i+1}]$$

Therefore in context Δ^{i+1} , the term $p^i[\psi^{i+1}]$ has type:

$$s^i[\psi^{i+1}] \rightarrow \Theta_{\mathbf{p}}^i[\text{in}^- \circ \sigma^{i+1} \circ \psi^{i+1}]$$

Furthermore, by equation (1), and Lemmas A.7 and A.5, since $w^i \notin \text{supp}(\Theta_{\mathbf{p}}^i[\psi^i])$, the target of this type satisfies:

$$\begin{aligned} \Theta_{\mathbf{p}}^i[\text{in}^- \circ \sigma^{i+1} \circ \psi^{i+1}] &= \Theta_{\mathbf{p}}^i[\text{in}^- \circ (\psi^i \uparrow w^i) \circ \tau^{i+1}] \\ &= \Theta_{\mathbf{p}}^i[\psi^i \circ \text{in}_{\Delta^i}^- \circ \tau^{i+1}] \\ &= \Theta_{\mathbf{p}}^i[\psi^i \circ \tau^{i+1}] \end{aligned}$$

Similarly, one can show that:

$$\Delta^{i+1} \vdash q^i[\psi^{i+1}] : \Theta_{\mathbf{p}}^i[\text{in}^+ \circ \sigma^{i+1}] \rightarrow t^i[\psi^{i+1}]$$

Finally, consider padding data $\Theta_{\mathbf{p}}$ associated to \mathbf{p} . We show that $\Theta_{\mathbf{p}}^i[\psi^i]$ satisfies the defining formula (\dagger), using (1) and Lemma A.8:

$$\begin{aligned} &\Theta_{\mathbf{p}}^{i+1}[\psi^{i+1}] \\ &= (p^i *_i (\Theta_{\mathbf{p}}^i \uparrow v^i)[\sigma^{i+1}] *_i q^i)[\psi^{i+1}] \\ &= p^i[\psi^{i+1}] *_i (\Theta_{\mathbf{p}}^i \uparrow v^i)[\sigma^{i+1} \circ \psi^{i+1}] *_i q^i[\psi^{i+1}] \\ &= p^i[\psi^{i+1}] *_i (\Theta_{\mathbf{p}}^i \uparrow v^i)[(\psi^i \uparrow w^i) \circ \tau^{i+1}] *_i q^i[\psi^{i+1}] \\ &= p^i[\psi^{i+1}] *_i (\Theta_{\mathbf{p}}^i[\psi^i] \uparrow w^i)[\tau^{i+1}] *_i q^i[\psi^{i+1}] \quad \square \end{aligned}$$

Lemma 3.9. *If Γ is a filtration, then so is $\Sigma\Gamma$. If the type family \mathbf{A} is adapted to Γ , then $\Sigma\mathbf{A}$ is adapted to $\Sigma\Gamma$, and if \mathbf{p} is padding data for \mathbf{A} , then $\Sigma\mathbf{p}$ is padding data for $\Sigma\mathbf{A}$, with associated padding $\Sigma\Theta_{\mathbf{p}}$:*

Proof. First we show that $\Sigma\Gamma$ is a filtration. The variable v^{i-1} is a maximal dimension variable in $\Sigma\Gamma^{i-1}$ which is of dimension i . By Lemma A.9, we have:

$$\Sigma(\Gamma^{i-1} \uparrow v^{i-1}) = (\Sigma\Gamma^{i-1}) \uparrow v^{i-1}$$

The substitution $\Sigma\sigma^i : \Sigma\Gamma^i \rightarrow (\Sigma\Gamma^{i-1}) \uparrow v^{i-1}$ satisfies, by definition of the suspension of substitutions:

$$\overrightarrow{v^{i-1}}[\Sigma\sigma^i] = v^i$$

Let $\mathbf{A} = (A^i = s^{i-1} \rightarrow t^{i-1})$ a type family adapted to Γ . We show that the type family $\Sigma\mathbf{A}$ is adapted to $\Sigma\Gamma$. Since in context Γ^m the variable v^m has type A^m , in context $\Sigma\Gamma^m$, the same variable has type ΣA^m . Moreover, in context

Γ^{i+1} , the terms s^i and t^i have type $A^i \llbracket \sigma^{i+1} \rrbracket$, so by Lemma A.1, in context $\Sigma \Gamma^{i+1}$, the terms Σs^i and Σt^i have type:

$$(\Sigma A^i) \llbracket \Sigma \sigma^{i+1} \rrbracket$$

Finally, consider a padding $\Theta_{\mathbf{p}}$ associated to \mathbf{p} , we show that $\Sigma \Theta_{\mathbf{p}}$ is a padding associated to \mathbf{p} . First, we note that in context Γ^{i+1} , the term p^i has type:

$$s^i \rightarrow \Theta_{\mathbf{p}}^i \llbracket \text{in}^- \circ \sigma^{i+1} \rrbracket$$

Thus, by Lemmas A.1 and A.9, the term Σp^i has the following type in context $\Sigma \Gamma^{i+1}$:

$$\Sigma s^i \rightarrow (\Sigma \Theta_{\mathbf{p}}^i) \llbracket \text{in}^- \circ \Sigma \sigma^{i+1} \rrbracket$$

We now check that $\Sigma \Theta_{\mathbf{p}}^i$ satisfies (\dagger) for $\Sigma \mathbf{p}$. Using Lemmas A.2, A.1 and A.10, we have:

$$\begin{aligned} \Sigma \Theta_{\mathbf{p}}^{i+1} &= \Sigma(p^i *_{i-1} (\Theta_{\mathbf{p}}^i \uparrow v^i) \llbracket \sigma^{i+1} \rrbracket *_{i-1} q^i) \\ &= \Sigma p^i *_{i+1} (\Sigma(\Theta_{\mathbf{p}}^i \uparrow v^i)) \llbracket \Sigma \sigma^{i+1} \rrbracket *_{i+1} \Sigma q^i \\ &= \Sigma p^i *_{i+1} (\Sigma \Theta_{\mathbf{p}}^i \uparrow v^i) \llbracket \Sigma \sigma^{i+1} \rrbracket *_{i+1} \Sigma q^i \quad \square \end{aligned}$$

Recall the definition of iterated functorialisation $t \uparrow^k v^i$ from the proof of Proposition 3.12:

$$\begin{aligned} t \uparrow^0 v^i &:= t \\ t \uparrow^{k+1} v^i &:= ((t \uparrow^k v^i) \uparrow v^{i+k}) \llbracket \sigma^{i+k+1} \rrbracket \end{aligned}$$

We now prove the following lemma.

Lemma C.1. *For any $m < i \leq n$ and $0 \leq j \leq n - i$, the unbiased padding satisfies the following equation:*

$$\Theta_{k,l}^i \uparrow^j v^i = p_{k,l}^{i-1} *_{i-1} (\Theta_{k,l}^{i-1} \uparrow^{j+1} v_l^{i-1}) *_{i-1} q_{k,l}^{i-1}$$

Proof. By induction on j . When $j = 0$, using the defining formula (\dagger) , we have:

$$\begin{aligned} \Theta_{k,l}^i \uparrow^0 v^n &= \Theta_{k,l}^i \\ &= p_{k,l}^{i-1} *_{i-1} (\Theta_{k,l}^{i-1} \uparrow v_l^{i-1}) \llbracket \sigma^i \rrbracket *_{i-1} q_{k,l}^{i-1} \\ &= p_{k,l}^{i-1} *_{i-1} (\Theta_{k,l}^{i-1} \uparrow^1 v_l^{i-1}) *_{i-1} q_{k,l}^{i-1} \end{aligned}$$

When $j > 0$, we have, by induction and using that $v^{i+j-1} \notin \text{supp}(p_{k,l}^{i-1})$ or $\text{supp}(q_{k,l}^{i-1})$, so these are fixed by σ^{i+j} :

$$\begin{aligned} \Theta_{k,l}^i \uparrow^j v^i &= ((\Theta_{k,l}^i \uparrow^{j-1} v^i) \uparrow v^{i+j-1}) \llbracket \sigma^{i+j} \rrbracket \\ &= ((p_{k,l}^{i-1} *_{i-1} (\Theta_{k,l}^{i-1} \uparrow^j v_l^{i-1}) *_{i-1} q_{k,l}^{i-1}) \uparrow v^{i+j-1}) \llbracket \sigma^{i+j} \rrbracket \\ &= p_{k,l}^{i-1} *_{i-1} ((\Theta_{k,l}^{i-1} \uparrow^j v_l^{i-1}) \uparrow v^{i+j-1}) \llbracket \sigma^{i+j} \rrbracket *_{i-1} q_{k,l}^{i-1} \\ &= p_{k,l}^{i-1} *_{i-1} (\Theta_{k,l}^{i-1} \uparrow^{j+1} v_l^{i-1}) *_{i-1} q_{k,l}^{i-1} \quad \square \end{aligned}$$

D Self-Duality of Unbiased Padding

This appendix is dedicated to the proof of Lemma 3.3.

Lemma D.1. *Let $0 \leq l$ and $i > l$. The contexts Γ_l^i appearing in the unbiased filtration satisfy, for any $M \subseteq \mathbb{N}_{>0}$:*

$$(\Gamma_l^i)^{\text{op } M} = \Gamma_l^i$$

Proof. It suffices to show the following

$$(I_l^{i-1})^{\text{op } M} = I_l^{i-1} :$$

By Lemma A.4, we have:

$$\begin{aligned} (I_l^{i-1})^{\text{op } M} &= ((\text{id}_x^{i-1})^{*l} \rightarrow (\text{id}_x^{i-1})^{*l})^{\text{op } M} \\ &= ((\text{id}_x^{i-1})^{*l})^{\text{op } M} \rightarrow ((\text{id}_x^{i-1})^{*l})^{\text{op } M} \\ &= \text{id}_x^{i-1} *_l \text{id}_x^{i-1} \quad \square \end{aligned}$$

Lemma D.2. *Let $0 \leq k, l$ and let $m := \min\{k, l\} + 1$. For any r , and for any $m \leq i$:*

$$(\Theta_{k,l}^i)^{\text{op}\{r\}} = \Theta_{k,l}^i \quad (\text{a})$$

Furthermore, for $i > m$:

$$\begin{aligned} (p_{k,l}^{i-1})^{\text{op}\{r\}} &= \begin{cases} p_{k,l}^{i-1} & r \neq i \\ q_{k,l}^{i-1} & r = i \end{cases} \\ (q_{k,l}^{i-1})^{\text{op}\{r\}} &= \begin{cases} q_{k,l}^{i-1} & r \neq i \\ p_{k,l}^{i-1} & r = i \end{cases} \quad (\text{b}) \end{aligned}$$

Proof. We proceed by induction on i .

When $i = m$, then have the equality on the terms $v_l^i = \Theta_{k,l}^i = (\Theta_{k,l}^i)^{\text{op } r}$.

For $i > m$, we first show (b). Recall the definitions:

$$\begin{aligned} \mathbb{P} &:= (x : \star) \\ p_{k,l}^{i-1} &:= \text{coh}(\mathbb{P} : (\text{id}_x^{i-1})^{*k} \rightarrow \Theta_{k,l}^{i-1} \llbracket (\text{id}_x^{i-1})^{*l} \rrbracket [x]) \\ q_{k,l}^{i-1} &:= \text{coh}(\mathbb{P} : \Theta_{k,l}^{i-1} \llbracket (\text{id}_x^{i-1})^{*l} \rrbracket \rightarrow (\text{id}_x^{i-1})^{*k} [x]) \end{aligned}$$

The context \mathbb{P} satisfies $\mathbb{P} = \mathbb{P}^{\text{op}\{r\}} = \mathbb{P}'$ and $\gamma_{\mathbb{P}} = \text{id}_{\mathbb{P}}$. We carry out the computation of $(p_{k,l}^{i-1})^{\text{op}\{r\}} \llbracket x \rrbracket$, The one of $(q_{k,l}^{i-1})^{\text{op}\{r\}} \llbracket x \rrbracket$ being similar. If $n \neq i$, we have, by induction and Lemmas A.4 and A.3:

$$\begin{aligned} &(p_{k,l}^{i-1})^{\text{op}\{r\}} \\ &= \text{coh}(\mathbb{P} : ((\text{id}_x^{i-1})^{*k})^{\text{op}\{r\}} \rightarrow (\Theta_{k,l}^{i-1} \llbracket (\text{id}_x^{i-1})^{*l} \rrbracket)^{\text{op}\{r\}} [x]) \\ &= \text{coh}(\mathbb{P} : (\text{id}_x^{i-1})^{*k} \rightarrow (\Theta_{k,l}^{i-1})^{\text{op}\{r\}} \llbracket (\text{id}_x^{i-1})^{*l} \rrbracket [x]) \\ &= \text{coh}(\mathbb{P} : (\text{id}_x^{i-1})^{*k} \rightarrow \Theta_{k,l}^{i-1} \llbracket (\text{id}_x^{i-1})^{*l} \rrbracket [x]) \\ &= p_{k,l}^{i-1} \end{aligned}$$

And similarly, if $r = i$, we have:

$$\begin{aligned}
& (p_{k,l}^{i-1})^{\text{op}\{r\}} \\
&= \text{coh}(\mathbb{P} : ((\Theta_{k,l}^{i-1} \llbracket (\text{id}_x^{i-1})^{*l} \rrbracket)^{\text{op}\{i\}} \rightarrow (\text{id}_x^{i-1})^{*k})^{\text{op}\{i\}})[x]) \\
&= \text{coh}(\mathbb{P} : (\Theta_{k,l}^{i-1})^{\text{op}\{r\}} \llbracket (\text{id}_x^{i-1})^{*l} \rrbracket \rightarrow (\text{id}_x^{i-1})^{*k})[x] \\
&= \text{coh}(\mathbb{P} : \Theta_{k,l}^{i-1} \llbracket (\text{id}_x^{i-1})^{*l} \rrbracket \rightarrow (\text{id}_x^{i-1})^{*k})[x] \\
&= q_{k,l}^{i-1}
\end{aligned}$$

We now show (a). We have:

$$\Theta_{k,l}^i = p_{k,l}^{i-1} *_{i-1} (\Theta_{k,l}^{i-1} \uparrow v_l^{i-1}) \llbracket \sigma^i \rrbracket *_{i-1} q_{k,l}^{i-1}$$

Denote the middle term $u := ((\Theta_{k,l}^{i-1} \uparrow v_l^{i-1}) \llbracket \sigma^i \rrbracket)$. We remark that by Lemma D.1, the substitution $\text{op}_{\Gamma_l^{i-1}, v_l^{i-1}, \{r\}}^\uparrow$ is the identity, and $(\sigma^i)^{\text{op}} = \sigma^i$. Then, induction, together with Lemmas A.3 and A.12, if we have:

$$\begin{aligned}
u^{\text{op}\{r\}} &= (\Theta_{k,l}^{i-1} \uparrow v_l^{i-1})^{\text{op}\{r\}} \llbracket \sigma^i \rrbracket \\
&= ((\Theta_{k,l}^{i-1})^{\text{op}\{r\}} \uparrow v_l^{i-1}) \llbracket \sigma^i \rrbracket \\
&= (\Theta_{k,l}^{i-1} \uparrow v_l^{i-1}) \llbracket \sigma^i \rrbracket \\
&= u
\end{aligned}$$

Using this equation and Lemma A.4 and the inductive hypothesis, for $r \neq i$ we have:

$$\begin{aligned}
(\Theta_{k,l}^i)^{\text{op}\{r\}} &= (p_{k,l}^{i-1})^{\text{op}\{r\}} *_{i-1} u^{\text{op}\{r\}} *_{i-1} (q_{k,l}^{i-1})^{\text{op}\{r\}} \\
&= p_{k,l}^{i-1} *_{i-1} u^{\text{op}\{r\}} *_{i-1} q_{k,l}^{i-1} \\
&= \Theta_{k,l}^i
\end{aligned}$$

Similarly, if $r = i$:

$$\begin{aligned}
(\Theta_{k,l}^i)^{\text{op}\{i\}} &= (q_{k,l}^{i-1})^{\text{op}\{i\}} *_{i-1} u^{\text{op}\{i\}} *_{i-1} (p_{k,l}^{i-1})^{\text{op}\{i\}} \\
&= p_{k,l}^{i-1} *_{i-1} u^{\text{op}\{i\}} *_{i-1} q_{k,l}^{i-1} \\
&= \Theta_{k,l}^i \quad \square
\end{aligned}$$

E Interchangers

This section is dedicated to the construction of the interchangers appearing in Proposition 3.12.

Construction E.1. We define a family of pasting contexts \mathbb{X}^n for $n \geq 2$. The pasting contexts \mathbb{X}^2 and \mathbb{X}^3 are illustrated in Fig. 14, and the general formula

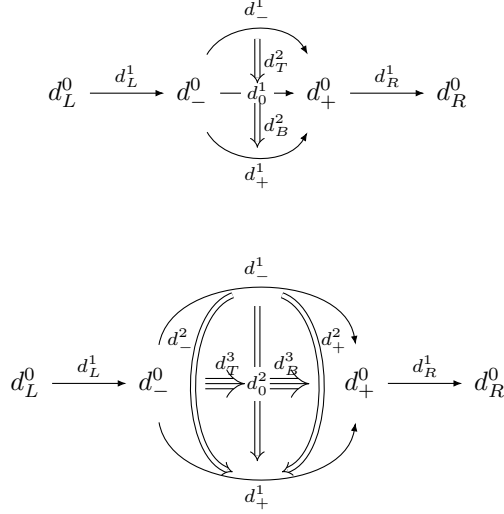


Figure 14: The pasting contexts \mathbb{X}^2 and \mathbb{X}^3

for \mathbb{X}^n is given by:

$$\left(\begin{array}{l} d_L^0, d_-^0 : \star, d_L^1 : d_L^0 \rightarrow d_-^0, d_+^0 : \star \\ d_-^1, d_+^1 : d_-^0 \rightarrow d_+^0, \dots, d_-^{n-2}, d_+^{n-2} : d_-^{n-3} \rightarrow d_+^{n-3}, \\ d_-^{n-1}, d_0^{n-1} : d_-^{n-2} \rightarrow d_+^{n-2}, d_T^n : d_-^{n-1} \rightarrow d_0^{n-1} \\ d_+^{n-1} : d_0^{n-2} \rightarrow d_+^{n-2}, d_B^n : d_0^{n-1} \rightarrow d_+^{n-1}, \\ d_R^0 : \star, d_R^1 : d_+^0 \rightarrow d_R^0 \end{array} \right)$$

In this context, given a term $\mathbf{X}^n \vdash t : A$ whose 0-dimensional source is d_-^0 and whose 0-target is d_+^0 , we write $w(t)$ for the whiskering $d_L^1 *_0 t *_0 d_R^1$. The interchangers appearing in Proposition 3.12 are then suspensions of the coherence:

$$\text{coh}(\mathbb{X}^n : w(d_T^n) *_{n-1} w(d_B^n) \rightarrow w(d_T^n *_{n-1} d_R^1))[\text{id}_{\mathbf{X}^n}]$$

We now construct the interchanger ζ^n appearing in Lemma 4.1.

Construction E.2. We define a family of pasting context \mathbb{Z}^n for $n \geq 2$, as the 0-gluing of two n -discs. The contexts \mathbb{Z}^2 and \mathbb{Z}^3 are illustrated in Fig. 15, and the general formula for \mathbb{Z}^n is:

$$\left(\begin{array}{l} d_L^0, d_0^0 : \star, d_{L-}^1, d_{L+}^1 : d_L^0 \rightarrow d_0^0, \\ d_{L-}^i, d_{L+}^i : d_{L-}^{i-1} \rightarrow d_{L+}^{i-1} \quad \text{for } 1 < i < n \\ d_L^n : d_{L-}^{n-1} \rightarrow d_{L+}^{n-1} \\ d_R^0 : \star, d_{R-}^1, d_{R+}^1 : d_0^0 \rightarrow d_R^0, \\ d_{R-}^i, d_{R+}^i : d_{R-}^{i-1} \rightarrow d_{R+}^{i-1} \quad \text{for } 1 < i < n \\ d_R^n : d_{R-}^{n-1} \rightarrow d_{R+}^{n-1} \end{array} \right)$$

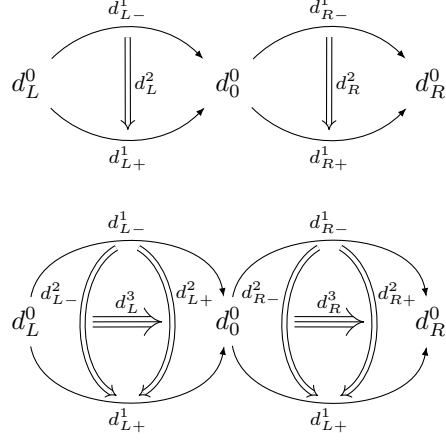


Figure 15: The pasting contexts \mathbb{Z}^2 and \mathbb{Z}^3

We then define the interchanger ζ^n to be the coherence:

$$\text{coh}(\mathbb{Z}^n : (d_L^n *_{\mathbf{0}} \text{id}_{d_{R-}^{n-1}}) *_{n-1} (\text{id}_{d_{L+}^{n-1}} *_{\mathbf{0}} d_R^n) \rightarrow d_L^n *_{\mathbf{0}} d_R^n) [\text{id}_{\mathbb{Z}^n}]$$

F Iterated Paddings

Lemma F.1. *Let $\Gamma = (\Gamma^i, v^i, \sigma^i)_{i=m}^n$ be a filtration, \mathbf{A} be a type adapted to Γ and \mathbf{p} padding data for \mathbf{A} . Suppose given a context Δ together with terms and substitutions*

$$\begin{array}{l} \Delta \vdash v : s \rightarrow_B t \qquad \Delta \vdash w : t \rightarrow_B u \\ \Delta \vdash \sigma_v : \Gamma \uparrow v^n \qquad \Delta \vdash \sigma_w : \Gamma \uparrow v^n \qquad \Delta \vdash \sigma_{v*w} : \Gamma^n \uparrow v^n \end{array}$$

such that

$$\vec{v}^n \llbracket \sigma_v \rrbracket = v \qquad \vec{v}^n \llbracket \sigma_w \rrbracket = w \qquad \vec{v}^n \llbracket \sigma_{v*w} \rrbracket = v *_{\mathbf{n}} w$$

and for every over variable x ,

$$x \llbracket \sigma_v \rrbracket = x \llbracket \sigma_w \rrbracket = x \llbracket \sigma_{v*w} \rrbracket.$$

Then, for any $0 \leq k \leq n - m$, there exists a term $\Xi_{\mathbf{p}}^{n-k \uparrow k+1}$ which is derivable in context Δ with type:

$$\begin{aligned} & ((\Theta^{n-k} \uparrow^k v^{n-k}) \uparrow v^n) \llbracket \sigma_v \rrbracket *_{\mathbf{n}} ((\Theta^{n-k} \uparrow^k v^{n-k}) \uparrow v^n) \llbracket \sigma_w \rrbracket \\ & \rightarrow ((\Theta^{n-k} \uparrow^k v^{n-k}) \uparrow v^n) \llbracket \sigma_{v *_{\mathbf{n}} w} \rrbracket \end{aligned}$$

Proof. The proof is exactly similar to that of Proposition 3.12. \square

Proposition 3.14. *For any filtration Γ and type family B adapted to it, the family $\Gamma_{/B}$ is a filtration. Moreover, a type family C is adapted to Γ if and only if it is adapted to $\Gamma_{/B}$.*

Proof. We first show that $\sigma_{/B}^i$ is well-typed. Denoting $B^i = s^{i-1} \rightarrow_{B^{i-1}[\sigma^i]} t^{i-1}$, we necessarily have the following, showing that $\sigma_{/B}^i$ is well-typed, and thus that $\Gamma_{/B}$ is a filtration:

$$(w^{i-1})^- \llbracket \sigma^i \rrbracket = s^{i-1} \qquad (w^{i-1})^+ \llbracket \sigma^i \rrbracket = t^{i-1}$$

For the second part, note that any type family $C = (C^i)_{i=m}^n$ adapted to either Γ or $\Gamma_{/B}$ must satisfy:

$$\Delta^i \vdash C^i$$

Since $A^m = B^m$, and since σ^i and $\sigma_{/B}^i$ coincide on Δ^{i-1} , C is adapted to Γ if and only if it is adapted to $\Gamma_{/B}$. \square

Proposition 3.15. *Given a filtration Γ with two types families B and C adapted to it. Suppose we have padding data $\mathbf{p} = (p_-^i, p_+^i)_{i=m}^{n-1}$ for B adapted to Γ and padding data $\mathbf{q} = (q_-^i, q_+^i)_{i=m}^{n-1}$ for C adapted to $\Gamma_{/B}$. Then there exists padding data $\mathbf{q} \square \mathbf{p} = (q_-^i \boxminus p_-^i, p_+^i \boxplus q_+^i)_{i=m}^{n-1}$ for C adapted to Γ and equivalences:*

$$\Gamma^i \vdash \mu_{\mathbf{q}, \mathbf{p}}^i : \Theta_{\mathbf{q}}^i \llbracket \Theta_{\mathbf{p}}^i \rrbracket \rightarrow \Theta_{\mathbf{q} \square \mathbf{p}}^i$$

Proof. Throughout this proof, we suppose that the filtration is given by $\Gamma = (\Gamma^i, v^i, \sigma^i)$ and we write:

$$A^i = a_-^{i-1} \rightarrow a_+^{i-1} \qquad B^i = b_-^{i-1} \rightarrow b_+^{i-1} \qquad C^i = c_-^{i-1} \rightarrow c_+^{i-1}$$

We write w^i for the chosen variable of $\Gamma_{/B}^i$ in the filtration $\Gamma_{/B}$. We construct my mutual the following:

- Padding data $\mathbf{q} \square \mathbf{p} = (q_-^i \boxminus p_-^i, p_+^i \boxplus q_+^i)_{i=m}^{n-1}$ for the type C adapted to the filtration Γ .
- Equivalences $\Gamma^i : \mu_{\mathbf{q}, \mathbf{p}}^i : \Theta_{\mathbf{q}}^i \llbracket \Theta_{\mathbf{p}}^i \rrbracket \rightarrow \Theta_{\mathbf{q} \square \mathbf{p}}^i$.

We first define $q_-^i \boxminus p_-^i$ and $p_+^i \boxplus q_+^i$ in terms of $\mu_{\mathbf{p}, \mathbf{q}}^i$:

$$\begin{aligned} q_-^i \boxminus p_-^i &:= q_-^i *_i (\Theta_{\mathbf{q}}^i \uparrow w^i) \llbracket p_-^i \rrbracket *_i \mu_{\mathbf{q}, \mathbf{p}}^i \llbracket \text{in}^- \circ \sigma^{i+1} \rrbracket \\ p_+^i \boxplus q_+^i &:= (\mu_{\mathbf{q}, \mathbf{p}}^i \llbracket \text{in}^+ \circ \sigma^{i+1} \rrbracket)^{-1} *_i (\Theta_{\mathbf{q}}^i \uparrow w^i) \llbracket p_+^i \rrbracket *_i q_+^i \end{aligned}$$

Thus we are left with defining the equivalences $\mu_{\mathbf{p}, \mathbf{q}}^i$. When $i = m$, it suffices to chose $\mu_{\mathbf{q}, \mathbf{p}}^m := \text{id}_{v^m}$. Now, supposing that $\mu_{\mathbf{q}, \mathbf{p}}^i$ has been constructed, we construct $\mu_{\mathbf{q}, \mathbf{p}}^{i+1}$ in three main steps. The source of $\mu_{\mathbf{q}, \mathbf{p}}^{i+1}$ is the following, writing $*$ for $*_i$:

$$\Theta_{\mathbf{q}}^{i+1} \llbracket \Theta_{\mathbf{p}}^{i+1} \rrbracket = q_-^i *_i (\Theta_{\mathbf{q}}^i \uparrow w^i) \llbracket p_-^i \rrbracket *_i (\Theta_{\mathbf{p}}^i \uparrow v^i) \llbracket \sigma^{i+1} \rrbracket *_i p_+^i *_i q_+^i$$

Our first step consist of a ternary variation of the pseudo-functoriality witness $\Xi_{\mathbf{q}}^{i\uparrow 1}$ defined in Lemma F.1. We can define this ternary variation X using associators and the binary one as follows:

$$\begin{aligned}
& (\Theta_{\mathbf{q}} \uparrow w^i) \llbracket p_-^i * (\Theta_{\mathbf{p}} \uparrow v^i) \llbracket \sigma^{i+1} \rrbracket * p_+^i \rrbracket \\
& \quad \downarrow (\text{associator}) \\
& (\Theta_{\mathbf{q}} \uparrow w^i) \llbracket (p_-^i * (\Theta_{\mathbf{p}} \uparrow v^i) \llbracket \sigma^{i+1} \rrbracket) * p_+^i \rrbracket \\
& \quad \downarrow \Xi_{\mathbf{q}}^{i\uparrow 1} \\
& (\Theta_{\mathbf{q}} \uparrow w^i) \llbracket p_-^i * (\Theta_{\mathbf{p}} \uparrow v^i) \llbracket \sigma^{i+1} \rrbracket \rrbracket * (\Theta_{\mathbf{q}} \uparrow w^i) \llbracket p_+^i \rrbracket \\
& \quad \downarrow \Xi_{\mathbf{q}}^{i\uparrow 1} \\
& ((\Theta_{\mathbf{q}} \uparrow w^i) \llbracket p_-^i \rrbracket * (\Theta_{\mathbf{q}} \uparrow w^i) \llbracket (\Theta_{\mathbf{p}} \uparrow v^i) \llbracket \sigma^{i+1} \rrbracket \rrbracket) * (\Theta_{\mathbf{q}} \uparrow w^i) \llbracket p_+^i \rrbracket \\
& \quad \downarrow (\text{associator}) \\
& (\Theta_{\mathbf{q}} \uparrow w^i) \llbracket p_-^i \rrbracket * (\Theta_{\mathbf{q}} \uparrow w^i) \llbracket (\Theta_{\mathbf{p}} \uparrow v^i) \llbracket \sigma^{i+1} \rrbracket \rrbracket * (\Theta_{\mathbf{q}} \uparrow w^i) \llbracket p_+^i \rrbracket
\end{aligned}$$

By Lemma A.8, the target of this cell is equal to:

$$(\Theta_{\mathbf{q}} \uparrow w^i) \llbracket p_-^i \rrbracket * (\Theta_{\mathbf{q}} \llbracket \Theta_{\mathbf{p}} \rrbracket \uparrow w^i) \llbracket \sigma^{i+1} \rrbracket * (\Theta_{\mathbf{q}} \uparrow w^i) \llbracket p_+^i \rrbracket$$

To proceed further, we use the term $\mu_{\mathbf{q},\mathbf{p}}^i \uparrow v^i$, which in context $\Gamma^i \uparrow v^i$ has type:

$$\mu_{\mathbf{q},\mathbf{p}}^i \llbracket \text{in}^- \rrbracket *_i (\Theta_{\mathbf{q}\square\mathbf{p}}^i \uparrow v^i) \rightarrow (\Theta_{\mathbf{q}} \llbracket \Theta_{\mathbf{p}} \rrbracket \uparrow v^i) *_i \mu_{\mathbf{q},\mathbf{p}}^i \llbracket \text{in}^+ \rrbracket$$

We now construct a new cell denoted Y , defined as a composite of 5 steps:

$$\begin{aligned}
& \Theta_{\mathbf{q}} \llbracket \Theta_{\mathbf{p}} \rrbracket \uparrow v^i \\
& \quad \downarrow (\text{unitor}) \\
& (\Theta_{\mathbf{q}} \llbracket \Theta_{\mathbf{p}} \rrbracket \uparrow v^i) *_i \text{id}(\Theta_{\mathbf{q}} \llbracket \Theta_{\mathbf{p}} \rrbracket \llbracket \text{in}^+ \rrbracket) \\
& \quad \downarrow ((\text{whiskering of } \varepsilon_{\mu_{\mathbf{q},\mathbf{p}}^i \llbracket \text{in}^+ \rrbracket}^{-1})) \\
& (\Theta_{\mathbf{q}} \llbracket \Theta_{\mathbf{p}} \rrbracket \uparrow v^i) *_i (\mu_{\mathbf{q},\mathbf{p}}^i \llbracket \text{in}^+ \rrbracket *_i (\mu_{\mathbf{q},\mathbf{p}}^i \llbracket \text{in}^+ \rrbracket)^{-1}) \\
& \quad \downarrow (\text{associator}) \\
& ((\Theta_{\mathbf{q}} \llbracket \Theta_{\mathbf{p}} \rrbracket \uparrow v^i) *_i \mu_{\mathbf{q},\mathbf{p}}^i \llbracket \text{in}^+ \rrbracket) *_i (\mu_{\mathbf{q},\mathbf{p}}^i \llbracket \text{in}^+ \rrbracket)^{-1} \\
& \quad \downarrow (\text{whiskering of } (\mu_{\mathbf{q},\mathbf{p}}^i \uparrow v^i)^{-1}) \\
& (\mu_{\mathbf{q},\mathbf{p}}^i \llbracket \text{in}^- \rrbracket *_i (\Theta_{\mathbf{q}\square\mathbf{p}}^i \uparrow v^i)) *_i (\mu_{\mathbf{q},\mathbf{p}}^i \llbracket \text{in}^+ \rrbracket)^{-1} \\
& \quad \downarrow (\text{associator}) \\
& \mu_{\mathbf{q},\mathbf{p}}^i \llbracket \text{in}^- \rrbracket *_i (\Theta_{\mathbf{q}\square\mathbf{p}}^i \uparrow v^i) *_i (\mu_{\mathbf{q},\mathbf{p}}^i \llbracket \text{in}^+ \rrbracket)^{-1}
\end{aligned}$$

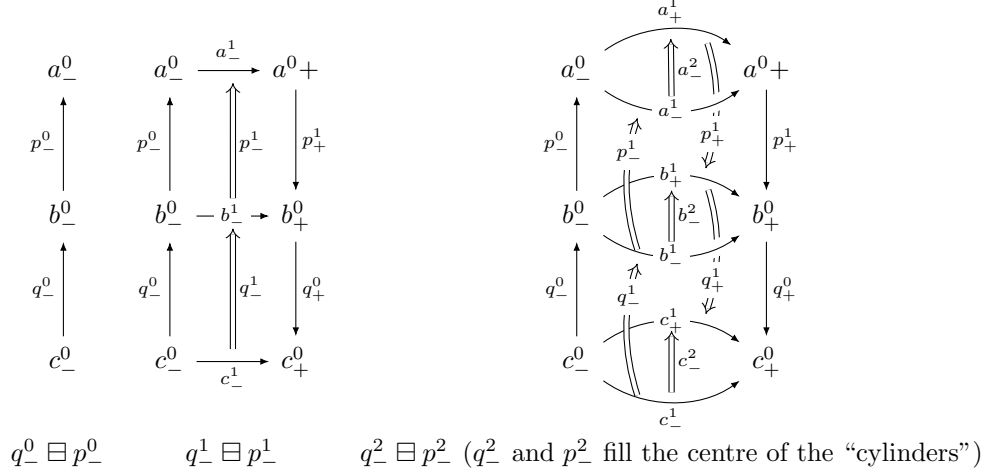


Figure 16: The constructions of $q^i \boxminus p^i$, for $i = 0, 1, 2$, where the filtration has height 0.

By Lemma B.3, the target of Y rewrites as:

$$\mu_{\mathbf{q}, \mathbf{p}}^i \llbracket \text{in}^- \circ \sigma^{i+1} \rrbracket *_i (\Theta_{\mathbf{q} \square \mathbf{p}}^i \uparrow v^i) *_i (\mu_{\mathbf{q}, \mathbf{p}}^i \llbracket \text{in}^+ \circ \sigma^{i+1} \rrbracket)^{-1}$$

This allows us to define the cell $\mu_{\mathbf{q}, \mathbf{p}}^{i+1}$ as a ternary composite as follows:

$$\begin{aligned}
 \tilde{p}_- &= (\Theta_{\mathbf{q}} \uparrow w^i) \llbracket p_-^i \rrbracket & \tilde{p}_+ &= (\Theta_{\mathbf{q}} \uparrow w^i) \llbracket p_+^i \rrbracket \\
 \mu_{\mathbf{q}, \mathbf{p}}^{i+1} &:= (q_-^i * (X *_{i+1} (\tilde{p}_- * Y * \tilde{p}_+)) * q_+^i) *_{i+1} (\text{associator})
 \end{aligned}$$

Here the associator is determined by the type:

$$\begin{aligned}
 &f_1 * (f_2 * (f_3 * f_4 * f_5) * f_6) * f_7 \\
 &\quad \downarrow \\
 &(f_1 * f_2 * f_3) * f_4 * (f_5 * f_6 * f_7)
 \end{aligned}$$

This construction is illustrated in Fig. 16. □