Duality for weak ω-categories

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Abstract

We define inductively the opposites of a weak globular ω-category with respect to a set of dimensions, and we show that the properties of being free on a globular set of a computad are preserved under forming opposites. We then provide a new description of hom ω-categories, and show that the opposites of a hom ω-category are hom ω-category of opposites of the original ω-category.

1 Introduction

In recent years, higher category theory has found applications in various fields. Globular higher categories and their computads have been used significantly in rewriting theory [20], homology theory [16], topological quantum field theory [4], and in the study of Homotopy Type Theory with equality [2, 18, 22]. Moreover, there has recently been significant progress [14] towards the conjecture that globular higher groupoids are equivalent to homotopy types [13].

Globular higher categories, or *ω-categories*, are a model of higher categories that have been introduced independently by a number of authors. Batanin and Leinster define them as algebras for some monad T on the category of globular sets [6, 17], Grothendieck and Maltsiniotis define them as models of some globular theory [19], while Finster and Mimram define them as models of some type theory [12]. The various notions of ω-category arising from those definitions have been shown to be equivalent by Ara [3], and by Benjamin, Finster and Mimram [7] respectively, so we may choose to work with either of them. In this paper, we will use Leinster's definition together with the recent description of the monad T by Dean et al. [11], which is in turn heavily inspired by the type-theoretic approach of Finster and Mimram. To obtain the monad T, Dean et al. first construct a category of computads Comp consisting of generating data for ω-categories, and then produce an adjunction

$$\mathsf{Free} : \mathsf{Glob} \rightleftarrows \mathsf{Comp} : \mathsf{Cell}$$

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between globular sets and computads inducing the monad T of Leinster.

Our primary contribution in this paper is the construction of the opposites of an ω-category. Such a construction is well-known in many contexts, including ordinary and enriched categories, bicategories, or $(\infty, 1)$-categories, and it has significantly helped in the development of the corresponding theories. The existence of dual categories allows for the unification of concepts, such as limits with colimits, or left and right Kan extensions and lifts, and for the simultaneous proof of theorems about them. That makes us believe that the existence of opposites for ω-categories will allow for the theory of ω-categories to be developed more easily.

Ordinary categories or $(\infty, 1)$-categories admit a single opposite, obtained by reversing the direction of all its arrows. Bicategories admit three opposites, obtained by reversing the direction of its arrows, its 2-cells, or both of them at the same time. More generally, it is expected that n-categories admit $2^n - 1$ opposites, obtained by reversing the direction of cells of certain dimensions. For ω-categories, we show that there exists a faithful action of the group

$$G = \mathcal{P}(\mathbb{N}_{>0})$$

of subsets of the positive natural numbers on the category of ω-categories, where a subset w acts on an ω-category by reversing the direction of its n-cells for every $n \in w$.

To define the opposite of an ω-category with respect to some $w \in G$, we first define the opposite of a globular set and a computad

$$\begin{aligned} \mathsf{op}_w &: \mathsf{Glob} \to \mathsf{Glob} \\ \mathsf{op}_w &: \mathsf{Comp} \to \mathsf{Comp} \end{aligned}$$

compatible with the adjunctions Free and Cell, in the sense that the following squares commute strictly and up to a natural isomorphism respectively:

$$\begin{tikzcd} \mathsf{Comp} \arrow[r, "\mathsf{op}_w"] & \mathsf{Comp} \\ \mathsf{Glob} \arrow[u, "\mathsf{Free}"] \arrow[r, "\mathsf{op}_w"] & \mathsf{Glob} \arrow[u, "\mathsf{Free}"'] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Comp} \arrow[r, "\mathsf{op}_w"] \arrow[d, "\mathsf{Cell}"'] & \mathsf{Comp} \arrow[d, "\mathsf{Cell}"] \\ \mathsf{Glob} \arrow[r, "\mathsf{op}_w"] \arrow[ur, phantom, "\overset{\mathsf{op}_w^{\mathsf{Cell}}}{\nearrow}"] & \mathsf{Glob} \end{tikzcd}$$

Out of this data, we get a natural isomorphism

$$\mathsf{op}_w^T : T\,\mathsf{op}_w \Rightarrow \mathsf{op}_w\,T$$

making the pair $(\mathsf{op}_w, \mathsf{op}_w^T)$ an automorphism of the monad T, hence an automorphism

$$\mathsf{op}_w : \omega\,\mathsf{Cat} \to \omega\,\mathsf{Cat}$$

of the category of ω-categories by the general theory of monads [21].

Our second goal is to show that the same techniques can give a novel construction of the hom ω-categories of an ω-category of Cottrell and Fujii [10].

Unilike strict ω-categories, the ω-categories we consider are not defined via enrichment, so it is not immediately clear that for everu ω-category X and objects $x, y \in X_0$, the globular set $\Omega(X, x, y)$ of cells with source x and target y admits the structure of an ω-category. Here we will give a new construction of such an ω-category structure. Starting from the suspension and hom adjunction

$$\Sigma : \mathsf{Glob} \rightleftarrows \mathsf{Glob}^{\star,\star} : \Omega$$

between globular sets and bipointed globular sets, we will extend the left adjoint to a functor

$$\Sigma : \mathsf{Comp} \to \mathsf{Comp}^{\star,\star}$$

between computads and bipointed computads, compatible with the $\mathsf{Free} \dashv \mathsf{Cell}$ adjunction, in that the following squares commute strictly and up to a natural transformation respectively

$$\begin{tikzcd} \mathsf{Comp} \arrow[r, "\Sigma"] & \mathsf{Comp}^{\star,\star} \\ \mathsf{Glob} \arrow[u, "\mathsf{Free}"] \arrow[r, "\Sigma"] & \mathsf{Glob}^{\star,\star} \arrow[u, "\mathsf{Free}^{\star,\star}"'] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Comp} \arrow[r, "\Sigma"] \arrow[d, "\mathsf{Cell}"'] & \mathsf{Comp}^{\star,\star} \arrow[d, "\mathsf{Cell}^{\star,\star}"] \\ \mathsf{Glob} \arrow[r, "\Sigma"] \arrow[ur, phantom, "\Sigma^{\mathsf{Cell}}"] & \mathsf{Glob}^{\star,\star} \end{tikzcd}$$

where $\mathsf{Free}^{\star,\star} \dashv \mathsf{Cell}^{\star,\star}$ are the obvious bipointed generalisations of Free and Cell respectively. Using the mate correspondence, we then get a natural transformation

$$\Omega^T : T\,\Omega \Rightarrow \Omega\,T^{\star,\star}$$

for $T^{\star,\star} = \mathsf{Cell}^{\star,\star}\,\mathsf{Free}^{\star,\star}$, which makes the pair (Ω, Ω^T) a morphism of monads from $T^{\star,\star}$ to T. On the level of algebras, this gives rise to a functor

$$\Omega : \omega\,\mathsf{Cat}^{\star,\star} \to \omega\,\mathsf{Cat}$$

from the category of bipointed ω-categories to the category of ω-categories, extending the hom functor Ω on globular sets defined above. Finally, we will show that the two constructions we describe in this article are related by the following commutative square

$$\begin{tikzcd} \omega\,\mathsf{Cat}^{\star,\star} \arrow[r, "\mathsf{op}_w"] \arrow[d, "\Omega"'] & \omega\,\mathsf{Cat}^{\star,\star} \arrow[d, "\Omega"] \\ \omega\,\mathsf{Cat} \arrow[r, "\mathsf{op}_{w-1}"'] & \omega\,\mathsf{Cat} \end{tikzcd}$$

where $w - 1 = \{x \in \mathbb{N}_{>0} : x + 1 \in w\}$.

2 Globular pasting diagrams

In this section, we briefly recall the notion of globular pasting diagrams, since they are a basic ingedient for any definition of weak ω-categories. Those are a family of globular sets such that diagrams indexed by them in a strict ω-category can be composed in a unique way. Pasting diagrams are parametrised by rooted, planar trees [6], an inductive description of whose as iterated lists was recently given by Dean et al [11, Section 2]. Our presentation in this section follows ibid. and Leinster [17, Appendix F.2], noting that pasting diagrams are bipointed globular sets generated by the suspension and the wedge sum operations.

To set the notation, we recall that *globular sets* are presheaves on the category $\mathbb{G}$ of globes with objects the natural numbers $\mathbb{N}$ and morphisms freely generated by the source and target inclusions

$$s_n, t_n \colon [n] \to [n+1]$$

under the globularity relations:

$$s_{n+1} \circ s_n = t_{n+1} \circ s_n \qquad\qquad s_{n+1} \circ t_n = t_{n+1} \circ t_n .$$

In other words, a globular set X consists of a set X_n for every natural number $n \in \mathbb{N}$ together with *source* and *target* functions

$$\mathsf{src}, \mathsf{tgt} \colon X_{n+1} \to X_n$$

satisfying the duals relations:

$$\mathsf{src} \circ \mathsf{src} = \mathsf{src} \circ \mathsf{tgt} \qquad\qquad \mathsf{tgt} \circ \mathsf{src} = \mathsf{tgt} \circ \mathsf{tgt} .$$

We will call elements of X_n the *n-cells of X*. The *k-source* and *k-target* of an n-cell x for $k < n$ are the k-cells defined by

$$\mathsf{src}_k\, x = \mathsf{src}(\cdots(\mathsf{src}\, x)) \qquad\qquad \mathsf{tgt}_k\, x = \mathsf{tgt}(\cdots(\mathsf{tgt}\, x))$$

We will denote by $\mathbb{D}^n$ the representable globular set associated to a natural number $n \in \mathbb{N}$, and call it the *n-disk*.

Bipointed globular sets are triples (X, x_-, x_+) consisting of a globular set and two distinguished 0-cells x_-, x_+ of it. They form a category $\mathsf{Glob}^{\star,\star}$ toether with morphisms of globular sets that preserve the distinguished 0-cells. By the Yoneda lemma, this is the coslice category ${}_{\mathbb{D}^0+\mathbb{D}^0}\backslash\, \mathsf{Glob}$ or the category of cospans of globular sets from $\mathbb{D}^0$ to itself.

The category of bipointed globular sets is locally finitely presentable as a coslice of a presheaf topos [1, Proposition 1.57], so in particular it is complete and cocomplete. Limits and connected colimits in $\mathsf{Glob}^{\star,\star}$ are computed as in Glob, i.e. they are created by the functor $\mathcal{U}_{\star,\star} \colon \mathsf{Glob}^{\star,\star} \to \mathsf{Glob}$ that forgets the basepoints. The coproduct of a family of bipointed sets is computed as the wide pushout of the corresponding maps out of $\mathbb{D}^0 + \mathbb{D}^0$.

Being a category of cospans, the category $\mathsf{Glob}^{\star,\star}$ is monoidal with respect to the composition of cospans, which we will call the *wedge sum*. More explicetely, the wedge sum $\vee$ of a pair of bipointed globular sets (X, x_-, x_+) and (Y, y_-, y_+)

is obtained by the following pushout square in Glob,

$$\begin{tikzcd}
 & & X \vee Y & & \\
 & X \arrow[ur, dashed, "\mathsf{in}_1"] & & Y \arrow[ul, dashed, "\mathsf{in}_2"'] & \\
\mathbb{D}^0 \arrow[ur, "x_-"] & & \mathbb{D}^0 \arrow[ul, "x_+"'] \arrow[ur, "y_-"] \arrow[uu, phantom, "\vee" very near end] & & \mathbb{D}^0 \arrow[ul, "y_+"']
\end{tikzcd}$$

with basepoints the image of x_- and the image of y_+ in the pushout. The unit of the wedge sum is given by the 0-disk $\mathbb{D}^0$ with both basepoints being its unique 0-cell. More generally, we will denote by $\bigvee_{i=1}^n X_i$ the iterated monoidal product of a finite family of bipointed globular sets $X_1, \ldots, X_n$, and we will denote the inclusion of the j-th component for $1 \leq j \leq n$ by

$$\mathsf{in}_j \colon X_j \to \bigvee_{i=1}^{n} X_j.$$

This is a morphism of globular sets, that is only a morphism of bipointed globular sets for $n = 1$.

The *suspension* of a globular set X is the bipointed globular set ΣX with two 0-cells v_- and v_+, and with positive dimensional cells given by

$$(\Sigma X)_{n+1} = X_n$$

for every $n \in \mathbb{N}$. The source and target maps of an n-cell of ΣX for $n > 2$ is given by its source and target in X, while the source and target of 1-cells are given by v_- and v_+ respectively. The basepoints of the suspensions are v_- and v_+. Suspension is left adjoint to the *path space* functor $\Omega \colon \mathsf{Glob}^{\star,\star} \to \mathsf{Glob}$ sending a bipointed globular set (X, x_-, x_+) to the globular set given by

$$\Omega(X, x_-, x_+)_n = \{x \in X_{n+1} \mid \mathsf{src}_0(x) = x_- \text{ and } \mathsf{tgt}_0(x) = x_+\}$$

The unit of the adjunction is the identity of the functor

$$\Omega\, \Sigma = \mathsf{id},$$

while the counit $\kappa : \Sigma\, \Omega \Rightarrow \mathsf{id}$ is the natural transformation with components the bipointed morphisms

$$\kappa : \Sigma\, \Omega(X, x_-, x_+) \to (X, x^-, x^+)$$

given by the subset inclusions $\Omega(X, x_-, x_+)_n \subseteq X_{n+1}$.

Finally, we have introduced all the ingredients to define globular pasting diagrams and the family parametrising them. We will call elements of that family Batanin trees following Dean et al [11].

Definition 1. A *Batanin tree* is a list $\mathsf{br}[B_1, \ldots, B_n]$, where the B_i are Batanin trees.

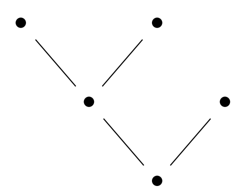
In other words, the set Bat of Batanin trees is the carrier of the initial algebra of the list endofunctor $\mathsf{List}\colon \mathsf{Set} \to \mathsf{Set}$ given by

$$\mathsf{List}\, X = \coprod_{n \in \mathbb{N}} X^n$$

with the obvious action on morphisms. In particular, there exists a tree $\mathsf{br}[]$ corresponding to the empty list, and using this tree, we can define more complicated trees, such as the tree

$$B = \mathsf{br}[\mathsf{br}[\mathsf{br}[], \mathsf{br}[]], \mathsf{br}[]].$$

It is convenient to visualise Batanin trees as planar trees by representing $\mathsf{br}[]$ as a tree with one root and no branches, and $\mathsf{br}[B_1, \ldots, B_n]$ as a tree with a new root and n branches, each of which is connected to the root of the tree corresponding to B_i. For example, the tree B above can be visualised as



The *dimension* of a Batanin tree is the height of the corresponding planar tree, or equivalently the maximum of the dimension of the cells in the corresponding globular pasting diagram, defined below. It can be computed recursively by

$$\mathsf{dim}(\mathsf{br}[B_1, \ldots, B_n]) = \mathsf{max}(\mathsf{dim}\, B_1 + 1, \ldots, \mathsf{dim}\, B_n + 1).$$

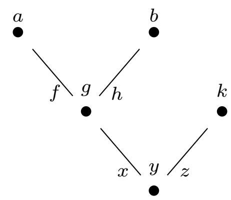
In particular, it follows that $\mathsf{br}[]$ is the unique tree of dimension 0.

Definition 2. The *bipointed globular set of positions* of a Batanin tree B is the bipointed globular set $\mathsf{Pos}^{\star,\star}(B)$ defined recursively by

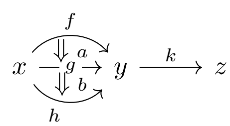
$$\mathsf{Pos}^{\star,\star}(\mathsf{br}[B_1 \ldots, B_n]) = \bigvee_{i=1}^{n} \Sigma\, \mathcal{U}_{\star,\star}\, \mathsf{Pos}^{\star,\star}(B_i).$$

The *globular pasting diagram* $\mathsf{Pos}(B)$ of a Batanin tree B is the underlying globular set of $\mathsf{Pos}^{\star,\star}(B)$, according to the fomulae in [17, Appendix F.2].

A way to calculate the globular set of positions of a tree is described in [8], where positions correspond to *sectors* of the tree, i.e. the spaces between two consecutive branches at each node, as well as the space before the first branch and the one after the last one. Under this description, the basepoint are given by the left-most and right-most sector at the root. For the tree B above, we can label the position as follows.



The dimension of a position is given by the distance of the node it is attached in from the root, while its source and target are given by the positions right below it. Therefore, the globular set of positions of B is the following globular set

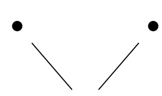


which is bipointed by the positions x and z respectively. Here, the positions f, g, h, a, b are the positions of the left branch of B, while k is the position of its right branch. The dimension of those positions has been raised by the suspension operation. The 0-positions x, y, z are the new cells created by the suspension operation. The two basepoints of $\mathsf{Pos}^{\star,\star}(B)$ are given by x and z.

Definition 3. The *k-boundary* of a Batanin tree B is the tree $\partial_k B$ defined recursively by

$$\begin{aligned} \partial_0 B &= \mathsf{br}[] \\ \partial_{k+1}\, \mathsf{br}[B_1, \ldots, B_n] &= \mathsf{br}[\partial_k B_1, \ldots, \partial_k B_n] \end{aligned}$$

The *k-boundary* of a tree B is the tree obrained by removing all nodes of B whose distance from the root is at least k. In terms of pasting diagrams, this amounts to removing all cells of dimension more than k and identifying all parallel k-cells. For example, the 1-boundary of the tree B considered above is the following tree.

$\partial_1 B =$  $\qquad \mathsf{Pos}(\partial_1 B) = \bullet \longrightarrow \bullet \longrightarrow \bullet$

The positions of the boundary can be included back into the positions of the original tree in two ways, the *source* and *target inclusions*

$$s_k^B, t_k^B \colon \mathsf{Pos}(\partial_k B) \to \mathsf{Pos}(B)$$

defined recursively as follows: the morphisms s_0^B and t_0^B out of $\mathbb{D}^0 \cong \mathsf{Pos}(\mathsf{br}[])$ select the first and second basepoint respectively, while for $B = \mathsf{br}[B_1, \ldots, B_n]$

the morphisms s^B_{k+1} and t^B_{k+1} are given by

$$s^B_{k+1} = \bigvee_{i=1}^{n} \Sigma\, s^{B_i}_k \qquad\qquad t^B_{k+1} = \bigvee_{i=1}^{n} \Sigma\, t^{B_i}_k.$$

In particular, the source and target inclusions are morphisms of bipointed globular sets when $k > 0$.

3 Computads and ω-categories

Dean et al. [11] recently presented a new definition of ω-categories and their computads, inspired by the type-theoretic definition of Finster and Mimram [12], and they showed that their notion of ω-category coincides with the operadic definition of Leinster [17]. In this approach, first a category of computads Comp is defined together with an adjunction

$$\mathsf{Free}\colon \mathsf{Glob} \rightleftarrows \mathsf{Comp}\colon \mathsf{Cell}$$

and then ω-categories are defined as algebras for the monad

$$T\colon \mathsf{Glob} \to \mathsf{Glob}$$

induced by the adjunction. We recall that morphisms of computads here are strict ω-functors, and not Batanin's morphisms of computads [5]. In other words, the comparison functor

$$K^T\colon \mathsf{Comp} \to \omega\,\mathsf{Cat}$$

is fully faithful and injective on objects.

We will briefly recall the definition of computads and the $\mathsf{Free} \dashv \mathsf{Cell}$ adjunction. First, categories Comp_n of n-computads are defined recursively for every natural number $n \in \mathbb{N}$, together with forgetful functor

$$u_n\colon \mathsf{Comp}_n \to \mathsf{Comp}_{n-1}$$

for $n > 0$. In the same mutual recursion, functors

$$\begin{aligned} \mathsf{Free}_n\colon &\ \mathsf{Glob} \to \mathsf{Comp}_n \\ \mathsf{Cell}_n\colon &\ \mathsf{Comp}_n \to \mathsf{Set} \\ \mathsf{Sphere}_n\colon &\ \mathsf{Comp}_n \to \mathsf{Set} \end{aligned}$$

are defined and natural transformations

$$\begin{aligned} \mathsf{bdry}_n\colon &\ \mathsf{Cell}_n \Rightarrow \mathsf{Sphere}_n\, u_n \\ \mathsf{pr}_i\colon &\ \mathsf{Sphere}_n \Rightarrow \mathsf{Cell}_n \end{aligned}$$

for $i = 1, 2$. Here the functors Cell_n and Sphere_n return the set of n-cells, and the set of pairs of parallel n-cells of the ω-category generated by a computad C, while bdry_n returns the source and the target of an n-cell.

An n-*computad* is a triple C consisting of an $(n-1)$-computad C_{n-1}, a set of n-dimensional *generators* V_n^C and an attaching function

$$\phi_n^C : V_n^C \to \mathsf{Sphere}_{n-1}(C_{n-1})$$

assigning to each generator a source and target. A morphism $\sigma\colon C \to D$ consists of a morphism $\sigma_{n-1}\colon C_{n-1} \to D_{n-1}$ and a function $\sigma_{n,V} : V_n^C \to \mathsf{Cell}_n D$ compatible with the source and target functions in the sense defined in [11, Section 3.1]. The forgetful functors u_n are the obvious projections. As a base case for this definition, here we let Comp_{-1} be the terminal category and Sphere_{-1} the functor choosing some terminal set.

The set $\mathsf{Cell}_n C$ of n-*cells of a computad* C is inductively defined together with the set of morphisms with target C and the function $\mathsf{bdry}_{n,C}$. Cells of C are either of the form $\mathsf{var}\, v$ for a generator $v \in V_n^C$, or when $n > 0$, they are *coherence cells* $\mathsf{coh}(B, A, \tau)$, where B is a tree of dimension at most n, A is an $(n-1)$-sphere of $\mathsf{Free}_{n-1} \mathsf{Pos}(B)$, satisfying a *fullness* condition that will be explained below, and $\tau\colon\ \mathsf{Free}_n \mathsf{Pos}(B) \to C$ is a morphism. The boundary of a cell is given recursively by the formula

$$\mathsf{bdry}_{n,C}(\mathsf{var}\, v) = \phi_n^C(v)$$
$$\mathsf{bdry}_{n,C}(\mathsf{coh}(B, A, \tau)) = \mathsf{Sphere}_{n-1}(\tau_{n-1})(A)$$

The functor Free_n sends a globular set X to the computad

$$\mathsf{Free}_n X = (\mathsf{Free}_{n-1} X, X_n, \phi_n^X) \qquad \phi_n^X(x) = (\mathsf{var}(\mathsf{src}\, x), \mathsf{var}(\mathsf{tgt}\, x)),$$

and a morphism $f\colon X \to Y$ to the morphism consisting of $\mathsf{Free}_{n-1} f$ and $\mathsf{var} \circ f_n$.

The functor Sphere_n sends an n-computad C to the set

$$\mathsf{Sphere}_n C = \{(a, b) \in \mathsf{Cell}_n C \times \mathsf{Cell}_n C \mid \mathsf{bdry}_n a = \mathsf{bdry}_n b\}$$

and acts on morphisms in the obvious way. The projection natural transformations are the obvious ones. We will denote by

$$\mathsf{src}, \mathsf{tgt}\colon\ \mathsf{Cell}_n \Rightarrow \mathsf{Cell}_{n-1} u_n$$

the composite of bdry_n with the projections.

The fullness condition mentioned above for $A = (a, b) \in \mathsf{Sphere}_n \mathsf{Free}_n \mathsf{Pos}(B)$ is a condition on the generators used to define a and b. It is equivalent to the statement that

$$a = \mathsf{Cell}_n \mathsf{Free}_n(s_n^B)(a') \qquad\qquad b = \mathsf{Cell}_n \mathsf{Free}_n(t_n^B)(b')$$

for cells a', b' of $\mathsf{Free}_n \mathsf{Pos}(\partial_n B)$ using all generators of $\partial_n B$. That means that the *support* of a', b' contains all positions of $\partial_n B$, where the support of an n-cell c over a computad C is the set of generators defined by

$$\mathsf{supp}(\mathsf{var}\, v) = \begin{cases} \{v\}, & \text{when } n = 0 \\ \{v\} \cup \mathsf{supp}(\mathsf{pr}_1\, \phi_n^C(v)) \cup \mathsf{supp}(\mathsf{pr}_2\, \phi_n^C(v)), & \text{when } n > 0 \end{cases}$$
$$\mathsf{supp}(\mathsf{coh}(B, A, \tau)) = \bigcup_{k \le n} \bigcup_{v \in \mathsf{Pos}_k(B)} \mathsf{supp}(\tau_{k,V}(v))$$

This completes the inductive definition. The category Comp of computads is the limit of the categories Comp_n for all $n \in \mathbb{N}$, i.e computads $C = (C_n)_{n\in\mathbb{N}}$ are sequences of n-computads C_n such that $u_{n+1}C_{n+1} = C_n$, and moprhisms of such are sequences of morphisms. The free functor

$$\mathsf{Free}\colon \mathsf{Glob} \to \mathsf{Comp}$$

is the functor with components Free_n for all $n \in \mathbb{N}$, while the cell functor

$$\mathsf{Cell}\colon \mathsf{Comp} \to \mathsf{Glob}$$

sends a computad C to the globular set consisting of $\mathsf{Cell}_n\, C_n$ for all $n \in \mathbb{N}$, and the source and target functions defined above. The unit of the adjunction

$$\eta\colon \mathsf{id} \Rightarrow \mathsf{Cell}\,\mathsf{Free}$$

sends a cell x of a globular set X to the generator $\mathsf{var}\, x$, while the counit

$$\varepsilon\colon \mathsf{Free}\,\mathsf{Cell} \Rightarrow \mathsf{id}$$

consists of the morphisms $\varepsilon_C\colon \mathsf{Free}\,\mathsf{Cell}\, C \to C$ given by the identities of the set

$$V_n^{\mathsf{Free}\,\mathsf{Cell}\, C} = \mathsf{Cell}_n\, C$$

for all $n \in \mathbb{N}$.

4 Opposites

An important feature of ordinary category theory is the duality stemming from the existence of opposite categories. This feature extends to higher categories, where we may define opposites by reversing the direction of all cells in certain dimensions. In this section, we will progressively define the opposite of a globular set, a computad, and an ω-category with respect to a set of dimensions $w \subseteq \mathbb{N}_{>0}$. We will then show that the formation of opposites in all those cases gives rise to an action of the Boolean group

$$G = \mathcal{P}(\mathbb{N}_{>0}) \cong \mathbb{Z}_2^{\mathbb{N}_{>0}}$$

of subsets of the positive integers with respect to symmetric difference. This group is clearly isomorphic to the group of functions $\mathbb{N}_{>0} \to \mathbb{Z}_2$ with pointwise multiplication, where each subset is identified with its indicator function. Abusing notation we will identify a subset w with its indicator function, and write $w(n)$ for the value of the indicator function at $n \in \mathbb{N}_{>0}$.

4.1 The opposite of a globular set

The group G acts on the category $\mathbb{G}$ of globes by swapping the source and target inclusions. More precisely, an element $w \in G$ acts as the identity-on-objects functor

$$\mathsf{op}_w\colon \mathbb{G} \to \mathbb{G}$$

given on the generating morphisms by

$$\mathsf{op}_w(s_n) = \begin{cases} t_n & \text{if } n+1 \in w, \\ s_n & \text{if } n+1 \notin w, \end{cases} \qquad \mathsf{op}_w(t_n) = \begin{cases} s_n & \text{if } n+1 \in w, \\ t_n & \text{if } n+1 \notin w. \end{cases}$$

The functor $\mathsf{op}_\emptyset$ is clearly the identity functor. Moreover, for every pair of elements $w, w' \in G$, we can easily check that

$$\mathsf{op}_w \mathsf{op}_{w'} = \mathsf{op}_{ww'}$$

so the assignment $w \mapsto \mathsf{op}_w$ is a group homomorphism $G \to \mathsf{Aut}(\mathbb{G})$. Since the group G is Abelian, this action extends to an action on the category Glob of globular sets by precomposition

$$\begin{gathered} \mathsf{op}\colon G \to \mathsf{Aut}(\mathsf{Glob}) \\ \mathsf{op}_w(X) = X \circ \mathsf{op}_w . \end{gathered}$$

The opposite $\mathsf{op}_w X$ of a globular set X therefore has the same cells as X, with the source and target of n-cells reversed for $n \in w$.

Since pasting diagrams are bipointed by their 0-source and 0-target inclusions, it will be useful to further extend this action to an action on bipointed globular sets

$$\mathsf{op}\colon G \to \mathsf{Aut}(\mathsf{Glob}^{\star,\star})$$

by letting op_w take a bipointed globular set (X, x_-, x_+) to the opposite globular set $\mathsf{op}_w X$ with the same basepoints when $1 \notin w$, and with the basepoints swapped otherwise.

Lemma 4. *For every $w \in G$, there exists a natural isomorphism*

$$\mathsf{op}^\Sigma_w\colon \Sigma \circ \ \mathsf{op}_{w-1} \Rightarrow \mathsf{op}_w \circ \Sigma$$

where $w-1 \in G$ is the sequence defined by $(w-1)(n) = w(n+1)$. Moreover, $\mathsf{op}^\Sigma_\emptyset$ is the identity natural transformation, and for every pair of elements $w, w' \in G$, the following diagram commutes:

$$\begin{tikzcd} \Sigma\, \mathsf{op}_{w-1} \mathsf{op}_{w'-1} \arrow[r, Rightarrow, "\mathsf{op}^\Sigma_w \mathsf{op}_{w'-1}"] \arrow[d, equal] & \mathsf{op}_w \Sigma\, \mathsf{op}_{w'-1} \arrow[r, Rightarrow, "\mathsf{op}_w \mathsf{op}^\Sigma_{w'}"] & \mathsf{op}_w \mathsf{op}_{w'} \Sigma \arrow[d, equal] \\ \Sigma\, \mathsf{op}_{ww'-1} \arrow[rr, Rightarrow, "\mathsf{op}^\Sigma_{ww'}"'] & & \mathsf{op}_{ww'} \Sigma \end{tikzcd}$$

Proof. For every globular set X, the bipointed globular sets $\Sigma\, \mathsf{op}_{w-1} X$ and $\mathsf{op}_w \Sigma X$ have the same sets of cells. Moreover, the source and target of an n-cell in both of them agree when $n > 2$: they are given by the target and source functions of X respectively when $n \in w$, and they are given by the source and target functions of X when $n \notin w$. The source and target of a 1-cell in the first one are given by v^- and v^+ respectively, while in the latter it is given by those when $1 \notin w$, and by v^+ and v^- when $1 \in w$. Therefore, we may define an isomorphism of globular sets

$$\mathsf{op}^{\Sigma}_{w,X} \colon\ \Sigma\, \mathsf{op}_{w-1} X \to \mathsf{op}_w \Sigma X$$

to be the identity on positive-dimensional cells, and to be given on 0-cells by

$$\mathsf{op}^{\Sigma}_{w,X}(v_{\pm}) = \begin{cases} \mathsf{op}^{\Sigma}_{w,X}(v_{\mp}), & \text{if } 1 \in w, \\ \mathsf{op}^{\Sigma}_{w,X}(v_{\pm}), & \text{if } 1 \notin w. \end{cases}$$

Since op_w reverses the basepoints if and only if $1 \in w$, we see that this is a morphism of bipointed globular sets. Naturality of these morphisms follows easily by the fact that it is the identity of positive-dimensional cells. Finally, the claimed diagram commutes for $w, w' \in G$: both morphisms are identity on positive-dimensional cells, they are the identity on 0-cells when $1 \in w \cap w'$ or $1 \notin w \cup w'$, and they swap the two 0-cells otherwise. $\square$

Lemma 5. *For every $w \in G$ and $n \in \mathbb{N}$, there exists a natural isomorphism*

$$\mathsf{op}^{\vee}_w \colon \bigvee_{i=1}^{n} \circ\ \mathsf{swap}_{w(1)} \circ (\mathsf{op}_w)^n \Rightarrow \mathsf{op}_w \circ \bigvee_{i=1}^{n}$$

where swap_0 *is the identity of* $(\mathsf{Glob}^{\star,\star})^n$*, while* swap_1 *is the automorphism*

$$\mathsf{swap}_1(X_1, \ldots, X_n) = (X_n, \ldots, X_1).$$

Moreover, $\mathsf{op}^{\vee}_{\emptyset}$ *is the identity natural transformation, and for every pair of elements $w, w' \in G$, the following diagram commutes:*

$$\begin{tikzcd}
\bigvee \circ (\mathsf{op}_{ww'})^n \circ \mathsf{swap}_{ww'(1)} \arrow[r, "\mathsf{op}^{\vee}_{ww'}"] \arrow[d, equal] & \mathsf{op}_{ww'} \circ \bigvee \arrow[ddd, equal] \\
\bigvee \circ (\mathsf{op}_w)^n \circ (\mathsf{op}_{w'})^n \circ \mathsf{swap}_{w(1)} \circ \mathsf{swap}_{w'(1)} \arrow[d, equal] & \\
\bigvee \circ (\mathsf{op}_w)^n \circ \mathsf{swap}_{w(1)} \circ (\mathsf{op}_{w'})^n \circ \mathsf{swap}_{w'(1)} \arrow[d, "\mathsf{op}^{\vee}_w"] & \\
\mathsf{op}_w \circ \bigvee \circ (\mathsf{op}_{w'})^n \circ \mathsf{swap}_{w'(1)} \arrow[r, "\mathsf{op}_w(\mathsf{op}^{\vee}_{w'})"] & \mathsf{op}_w\, \mathsf{op}'_w \circ \bigvee
\end{tikzcd}$$

Proof. Fix $n \in \mathbb{N}$ and $w \in G$ and let $X_1, \ldots, X_n$ be bipointed globular sets and suppose first that $1 \notin w$, so that the basepoints of X_i and $\mathsf{op}_w X_i$ agree. The functor op_w on globular sets preserves $\mathbb{D}^0$, and it preserves colimits, being an equivalence of categories. Therefore, there exists a natural isomorphism of globular sets

$$\mathsf{op}^{\vee}_w \colon \bigvee_{i=1}^{n} (\mathsf{op}_w X_i) \to \mathsf{op}_w \left(\bigvee_{i=1}^{n} X_i \right),$$

that can be easily seen to preserve the basepoints. Moreover, since op_w preserves the cells of a globular set, and colimits of globular sets are computed pointwise, we may take $\mathsf{op}_w^{\vee}$ to be the identity.

Suppose now that $1 \in w$, so that the functor op_w swaps the basepoints. Using that op_w preserves colimits and $\mathbb{D}^0$, we see that $\mathsf{op}_w(\bigvee_{i=1}^n X_i)$ is the colimit of the following diagram.

$$\begin{tikzcd}
\mathsf{op}_w X_1 & & \cdots & & \mathsf{op}_w X_n \\
& \mathbb{D}^0 \arrow[ul, "x_{1,+}"'] \arrow[ur, "x_{2,-}"'] & & \mathbb{D}^0 \arrow[ul, "x_{n-1,+}"'] \arrow[ur, "x_{n,-}"'] &
\end{tikzcd}$$

On the other hand, $\bigvee_{i=n}^1 \mathsf{op}_w X_i$ is the colimit of the following diagram:

$$\begin{tikzcd}
\mathsf{op}_w X_n & & \cdots & & \mathsf{op}_w X_1 \\
& \mathbb{D}^0 \arrow[ul, "x_{n,-}"'] \arrow[ur, "x_{n-1,+}"'] & & \mathbb{D}^0 \arrow[ul, "x_{2,-}"'] \arrow[ur, "x_{1,+}"'] &
\end{tikzcd}$$

By symmetry of pushouts, we get a natural isomorphism of globular sets

$$\mathsf{op}_w^{\vee} \colon \bigvee_{i=n}^{1} (\mathsf{op}_w X_i) \to \mathsf{op}_w \left(\bigvee_{i=1}^{n} X_i \right)$$

that can be easily seen to preserve the basepoints. Since colimits are computed object-wise, this isomorphism is given level-wise by the symmetry of pushouts.

Knowing how those isomorphisms are defined pointwise, we can easily deduce that the claimed diagram commutes for every pair $w, w' \in G$. If $1 \notin w \cup w'$, then both sides of the diagram are identities. If $1 \in w \cap w'$ again both are identities, since the symmetry of the pushout squares to the identity. Finally, when $1 \in ww'$, then both sides are given by the symmetry of the pushout, so they agree. $\square$

Using those lemmas, we can deduce that pasting diagrams are closed under the formation of opposites: we define recursively on the Batanin tree B for every $w \in G$ the *w-opposite Batanin tree* $\mathsf{op}_w B$ by the formula

$$\mathsf{op}_w(\mathsf{br}[B_1, \ldots, B_n]) = \mathsf{br}(\mathsf{swap}_{w(1)}[\mathsf{op}_{w-1} B_1, \ldots, \mathsf{op}_{w-1} B_n]),$$

where swap_0 is the identity of the set of lists, while swap_1 reverses a list

$$\mathsf{swap}_1[B_1, \ldots, B_n] = [B_n, \ldots, B_1].$$

The opposite tree realizes the opposite pasting diagram, in the sense that there exists an isomorphism of bipointed globular sets

$$\mathsf{op}_w^B \colon \mathsf{Pos}^{\star,\star}(\mathsf{op}_w B) \to \mathsf{op}_w(\mathsf{Pos}^{\star,\star}(B)).$$

We can define this isomorphism recursively on $B = \mathsf{br}[B_1, \ldots, B_n]$ to be the following composite

$$\begin{aligned}
\mathsf{Pos}^{\star,\star}(\mathsf{op}_w\, B) &= \bigvee_{i=1}^{n} \mathsf{swap}_{w(1)}(\Sigma\, \mathsf{Pos}(\mathsf{op}_{w-1}\, B_i)) \\
&\xrightarrow{\bigvee \mathsf{swap}_{w(1)}\, \Sigma\, \mathsf{op}_{w-1}^{B_i}} \bigvee_{i=1}^{n} \mathsf{swap}_{w(1)}(\Sigma\, \mathsf{op}_{w-1}\, \mathsf{Pos}(B_i)) \\
&\xrightarrow{\bigvee \mathsf{swap}_{w(1)}\, \mathsf{op}_w^{\Sigma}} \bigvee_{i=1}^{n} \mathsf{swap}_{w(1)}(\mathsf{op}_w\, \Sigma\, \mathsf{Pos}(B_i)) \\
&\xrightarrow{\mathsf{op}_w^{\vee}} \mathsf{op}_w \left(\bigvee_{i=1}^{n} \Sigma\, \mathsf{Pos}(B_i) \right) = \mathsf{op}_w(\mathsf{Pos}^{\star,\star}(B)).
\end{aligned}$$

Lemma 6. *The isomorphism $\mathsf{op}_{\emptyset}^{B}$ is the identity for every tree B, and for any $w, w' \in G$, the following diagram of isomorphisms commutes:*

$$\begin{tikzcd}
\mathsf{Pos}(\mathsf{op}_w\, \mathsf{op}_{w'}\, B) \arrow[r, "\mathsf{op}_w^{\mathsf{op}_{w'}\, B}"] \arrow[d, equal] & \mathsf{op}_w(\mathsf{Pos}(\mathsf{op}_{w'}\, B)) \arrow[r, "\mathsf{op}_w(\mathsf{op}_{w'}^{B})"] & \mathsf{op}_w\, \mathsf{op}_{w'}\, \mathsf{Pos}(B) \arrow[d, equal] \\
\mathsf{Pos}(\mathsf{op}_{ww'}\, B) \arrow[rr, "\mathsf{op}_{ww'}^{B}"'] & & \mathsf{op}_{ww'}\, \mathsf{Pos}(B)
\end{tikzcd}$$

Proof. This lemma is an easy induction on B, using naturality of the isomorphisms in Lemmas 4 and 5, and of the commuting diagrams there. $\square$

Lemma 7. *For every $w \in G$, $k \in \mathbb{N}$ and Batanin tree B,*

$$\mathsf{op}_w\, \partial_k = \partial_k\, \mathsf{op}_w\, .$$

Moreover, the following equations hold

$k+1 \in w$	$k+1 \notin w$
$\mathsf{op}_w(t_k^B) \circ \mathsf{op}_w^{\partial_k B} = \mathsf{op}_w^B \circ\, s_k^{\mathsf{op}_w\, B}$	$\mathsf{op}_w(s_k^B) \circ \mathsf{op}_w^{\partial_k B} = \mathsf{op}_w^B \circ\, s_k^{\mathsf{op}_w\, B}$
$\mathsf{op}_w(s_k^B) \circ \mathsf{op}_w^{\partial_k B} = \mathsf{op}_w^B \circ\, t_k^{\mathsf{op}_w\, B}$	$\mathsf{op}_w(t_k^B) \circ \mathsf{op}_w^{\partial_k B} = \mathsf{op}_w^B \circ\, t_k^{\mathsf{op}_w\, B}$

Proof. We proceed by induction on k. For $k = 0$ both $\mathsf{op}_w\, \partial_k B$ and $\partial_k\, \mathsf{op}_w\, B$ are equal to the disk D_0, and the equations state that op_w^B preserves the basepoints. Suppose therefore that the result is true for some $k \in \mathbb{N}$ to prove that it also holds for $k+1$. Letting $B = \mathsf{br}[B_1, \ldots, B_n]$, we see that

$$\begin{aligned}
\mathsf{op}_w\, \partial_{k+1} B &= \mathsf{op}_w(\mathsf{br}[\partial_k B_1, \ldots, \partial_k B_n]) \\
&= \mathsf{br}(\mathsf{swap}_{w(1)}[\mathsf{op}_{w-1}\, \partial_k B_1, \ldots, \mathsf{op}_{w-1}\, \partial_k B_n]) \\
&= \mathsf{br}(\mathsf{swap}_{w(1)}[\partial_k\, \mathsf{op}_{w-1}\, B_1, \ldots, \partial_k\, \mathsf{op}_{w-1}\, B_n]) \\
&= \partial_{k+1}\, \mathsf{br}(\mathsf{swap}_{w(1)}[\mathsf{op}_{w-1}\, B_1, \ldots, \mathsf{op}_{w-1}\, B_n]) \\
&= \partial_{k+1}\, \mathsf{op}_w\, B
\end{aligned}$$

by the inductive hypothesis.

We will prove the first equation in the case that $k+1 \in w$ and $1 \in w$. The other equation and the rest of the cases follow by the same argument. By the inductive hypothesis, we may assume that for $1 \leq i \leq n$, the following square commutes

$$\begin{tikzcd}
\mathsf{Pos}(\mathsf{op}_{w-1}\, B_i) \arrow[r, "\mathsf{op}_{w-1}^{B_i}"] & \mathsf{op}_{w-1}(\mathsf{Pos}(B_i)) \\
\mathsf{Pos}(\partial_{k-1}\, \mathsf{op}_{w-1}\, B_i) \arrow[u, "s_{k-1}^{\mathsf{op}_{w-1}\, B_i}"] \arrow[d, equal] & \\
\mathsf{Pos}(\mathsf{op}_{w-1}\, \partial_{k-1} B_i) \arrow[r, "\mathsf{op}_{w-1}^{\partial_{k-1} B_i}"'] & \mathsf{op}_{w-1}\, \mathsf{Pos}(\partial_{k-1} B_i) \arrow[uu, "\mathsf{op}_{w-1}(t_{k-1}^{B_i})"']
\end{tikzcd}$$

Applying the suspension functor and then the wedge sum from n to 1, we get that the left square below commutes. Naturality of the isomorphisms in Lemmas 4 and 5 then imply that the right square below also commutes.

$$\begin{tikzcd}
\bigvee \Sigma\, \mathsf{Pos}(\mathsf{op}_{w-1}\, B_i) \arrow[r] & \bigvee \Sigma\, \mathsf{op}_{w-1}\, \mathsf{Pos}(B_i) \arrow[r] & \mathsf{op}_w \bigvee \Sigma\, \mathsf{Pos}(B_i) \\
\bigvee \Sigma\, \mathsf{Pos}(\partial_{k-1}\, \mathsf{op}_{w-1}\, B_i) \arrow[u, "\bigvee \Sigma\, s_{k-1}^{\mathsf{op}_{w-1}\, B_i}"'] \arrow[d, equal] & & \\
\bigvee \Sigma\, \mathsf{Pos}(\mathsf{op}_{w-1}\, \partial_{k-1} B_i) \arrow[r] & \bigvee \Sigma\, \mathsf{op}_{w-1}\, \mathsf{Pos}(\partial_{k-1} B_i) \arrow[r] \arrow[uu, "\bigvee \Sigma\, \mathsf{op}_{w-1}(t_{k-1}^{B_i})"] & \mathsf{op}_w \bigvee \Sigma\, \mathsf{Pos}(\partial_{k-1} B_i) \arrow[uu, "\mathsf{op}_w \bigvee \Sigma\, t_{k-1}^{B_i}"]
\end{tikzcd}$$

The outer part of the diagram though is precisely the square:

$$\begin{tikzcd}
\mathsf{Pos}(\mathsf{op}_w\, B) \arrow[r, "\mathsf{op}_w^B"] & \mathsf{op}_w(\mathsf{Pos}(B)) \\
\mathsf{Pos}(\partial_k\, \mathsf{op}_w\, B) \arrow[u, "s_k^{\mathsf{op}_w\, B}"] \arrow[d, equal] & \\
\mathsf{Pos}(\mathsf{op}_w\, \partial_k B) \arrow[r, "\mathsf{op}_w^{\partial_k B}"'] & \mathsf{op}_w\, \mathsf{Pos}(\partial_k B) \arrow[uu, "\mathsf{op}_w(t_k^B)"']
\end{tikzcd}$$

whose commutativity amounts to the first equation. $\square$

4.2 The opposite of a computad

The opposite of a computad is defined similarly to the opposite of a globular set by swapping the source and target of its generators. To define this action, we fix an element $w \in G$ and define recursively on the dimension $n \in \mathbb{N}$, an endofunctor and two natural transformation

$$\begin{aligned} \mathsf{op}_w \colon \; & \mathsf{Comp}_n \to \mathsf{Comp}_n \\ \mathsf{op}_w^{\mathsf{Cell}} \colon \; & \mathsf{Cell}_n \Rightarrow \mathsf{Cell}_n \, \mathsf{op}_w \\ \mathsf{op}_w^{\mathsf{Sphere}} \colon \; & \mathsf{Sphere}_n \Rightarrow \mathsf{Sphere}_n \, \mathsf{op}_w \end{aligned}$$

satisfying the following properties:

(OP1) forming opposites commutes with the forgetful functors, and the inclusion of globular sets into computads

$$\begin{tikzcd} \mathsf{Comp}_{n+1} \arrow[r, "\mathsf{op}_w"] \arrow[d, "u_{n+1}"'] & \mathsf{Comp}_{n+1} \arrow[d, "u_{n+1}"] \\ \mathsf{Comp}_n \arrow[r, "\mathsf{op}_w"'] & \mathsf{Comp}_n \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Glob} \arrow[r, "\mathsf{op}_w"] \arrow[d, "\mathsf{Free}_n"'] & \mathsf{Glob} \arrow[d, "\mathsf{Free}_n"] \\ \mathsf{Comp}_n \arrow[r, "\mathsf{op}_w"'] & \mathsf{Comp}_n \end{tikzcd}$$

(OP2) the natural transformations are compatible with the boundary natural transformation

$$\begin{tikzcd} \mathsf{Cell}_{n+1} \arrow[rr, Rightarrow, "\mathsf{op}_w^{\mathsf{Cell}}"] \arrow[d, Rightarrow, "\mathsf{bdry}_{n+1}"'] & & \mathsf{Cell}_{n+1} \, \mathsf{op}_w \arrow[d, Rightarrow, "\mathsf{bdry}_{n+1} \, \mathsf{op}_w"] \\ \mathsf{Sphere}_n \, u_{n+1} \arrow[r, Rightarrow, "\mathsf{op}_w^{\mathsf{Sphere}} \, u_{n+1}"'] & \mathsf{Sphere}_n \, \mathsf{op}_w \, u_{n+1} \arrow[r, equal] & \mathsf{Sphere}_n \, u_{n+1} \, \mathsf{op}_w \end{tikzcd}$$

(OP3) the natural transformation $\mathsf{op}_w^{\mathsf{Sphere}}$ swaps the two cells of a sphere when $n+1 \in w$ and leaves them unchanged otherwise, in the sense that the following diagrams commute for $i = 1, 2$

$$\begin{tikzcd} \mathsf{Sphere}_n \arrow[r, Rightarrow, "\mathsf{op}_w^{\mathsf{Sphere}}"] \arrow[d, Rightarrow, "\mathsf{pr}_i"'] \arrow[dr, phantom, "n+1 \notin w"] & \mathsf{Sphere}_n \, \mathsf{op}_w \arrow[d, Rightarrow, "\mathsf{pr}_i"] \\ \mathsf{Cell}_n \arrow[r, Rightarrow, "\mathsf{op}_w^{\mathsf{Cell}}"'] & \mathsf{Cell}_n \, \mathsf{op}_w \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Sphere}_n \arrow[r, Rightarrow, "\mathsf{op}_w^{\mathsf{Sphere}}"] \arrow[d, Rightarrow, "\mathsf{pr}_i"'] \arrow[dr, phantom, "n+1 \in w"] & \mathsf{Sphere}_n \, \mathsf{op}_w \arrow[d, Rightarrow, "\mathsf{pr}_{2-i}"] \\ \mathsf{Cell}_n \arrow[r, Rightarrow, "\mathsf{op}_w^{\mathsf{Cell}}"'] & \mathsf{Cell}_n \, \mathsf{op}_w \end{tikzcd}$$

(OP4) the natural transformation $\mathsf{op}_w^{\mathsf{Cell}}$ preserves generators, in that for every globular set X and $x \in X_n$, we have that

$$\mathsf{op}_{w,\mathsf{Free}\,X}^{\mathsf{Cell}}(\mathsf{var}\, x) = \mathsf{var}\, x.$$

(OP5) the natural transformation $\mathsf{op}^{\mathsf{Sphere}}$ preserves fullness, in that for every full n-sphere A of $\mathsf{Free}\,\mathsf{Pos}(B)$, the n-sphere

$$A' = \mathsf{Sphere}_n\, \mathsf{Free}_n(\mathsf{op}_w^B)^{-1}(\mathsf{op}_w^{\mathsf{Sphere}}(A))$$

of $\mathsf{Free}\,\mathsf{Pos}(\mathsf{op}_w\, B)$ is also full.

As a base case, we define op_w and $\mathsf{op}_w^{\mathsf{Sphere}}$ to be the identities of Comp_{-1} and Sphere_{-1} respectively. Let therefore $n \in \mathbb{N}$ and suppose inductively that data as above has been defined for all natural numbers less than n, satisfying the given properties.

Computads. First we will define the action of op_w on all n-computads. Let $C = (C_{n-1}, V_n^C, \phi_n^C)$ be an n-computad. The opposite computad $\mathsf{op}_w\, C$ consists of the opposite computad $\mathsf{op}_w\, C_{n-1}$, the same set of generators V_n^C, and the attaching function

$$\phi_n^{\mathsf{op}_w\, C} \colon V_n^C \xrightarrow{\phi_n^C} \mathsf{Sphere}_{n-1}\, C_{n-1} \xrightarrow{\mathsf{op}_w^{\mathsf{Sphere}}} \mathsf{Sphere}_{n-1}(\mathsf{op}_w\, C_{n-1}).$$

By Properties (OP3) and (OP4), we can easily deduce that op_w commutes with the inclusion Free_n on objects, while it clearly commutes with the forgetful functors u_n by definition.

Cells and morphisms. We will then define op_w on morphisms σ of n-computads of target C, together with the component of the natural transformation $\mathsf{op}_w^{\mathsf{Cell}}$ at C mutually recursively. For a generator $v \in V_n^C$, we let

$$\mathsf{op}_w^{\mathsf{Cell}}(\mathsf{var}\, v) = \mathsf{var}\, v,$$

and we observe that

$$\mathsf{bdry}_n\, \mathsf{op}_w^{\mathsf{Cell}}(\mathsf{var}\, v) = \mathsf{op}_w^{\mathsf{Sphere}}\, \mathsf{bdry}_n(\mathsf{var}\, v).$$

Given a coherence cell $c = \mathsf{coh}(B, A, \tau)$ of C, we may assume that recursively that $\mathsf{op}_w(\tau)$ has been defined, and let

$$\begin{aligned} A' &= \mathsf{Sphere}_{n-1}\, \mathsf{Free}_{n-1}(\mathsf{op}_w^B)^{-1}(\mathsf{op}_w^{\mathsf{Sphere}}(A)) \\ \mathsf{op}_w^{\mathsf{Cell}}(c) &= \mathsf{coh}(\mathsf{op}_w\, B, A', \mathsf{op}_w(\tau) \circ \mathsf{Free}_n(\mathsf{op}_w^B)) \end{aligned}$$

We then observe again that the boundary of this cell is given by

$$\begin{aligned} \mathsf{bdry}_n\, \mathsf{op}_w^{\mathsf{Cell}}(c) &= \mathsf{Sphere}_{n-1}(\mathsf{op}_w\, \tau_{n-1})(\mathsf{op}_w^{\mathsf{Sphere}}\, A) \\ &= \mathsf{op}_w^{\mathsf{Sphere}}(\mathsf{Sphere}_{n-1}(\tau_{n-1})(A)) \\ &= \mathsf{op}_w^{\mathsf{Sphere}}\, \mathsf{bdry}_n(c). \end{aligned}$$

Finally, for a morphism $\sigma = (\sigma_{n-1}, \sigma_V) : D \to C$, we define assume that $\mathsf{op}_w^{\mathsf{Cell}}$ has been defined on cells of the form $\sigma_V(v)$ for $v \in V_n^D$ and define

$$\mathsf{op}_w(\sigma) = (\mathsf{op}_w\, \sigma_{n-1}, \mathsf{op}_w^{\mathsf{Cell}} \circ \sigma_V) : \mathsf{op}_w\, D \to \mathsf{op}_w\, C$$

This is a well-defined morphism of computads by the observation on the boundary of the cells $\mathsf{op}_w^{\mathsf{Cell}}(c)$, i.e. by Properties (OP2).

It follows immediately from the definition that op_w commutes with the forgetful functor u_n on morphisms as well. Using that $\mathsf{op}_w^{\mathsf{Cell}}$ preserves generators, we can also deduce that op_w commutes with the inclusion Free_n on morphisms as well. Therefore, we have shown Properties (OP1), (OP2) and (OP4) so far.

Naturality. We will now show that op_w is a functor and that $\mathsf{op}_w^{\mathsf{Cell}}$ is natural. For that, we fix a morphism of n-computads $\sigma \colon C \to D$, and we proceed recursively to show that the following square commutes

$$\begin{tikzcd}
\mathsf{Cell}_n\, C \arrow[r, "\mathsf{Cell}_n\, \sigma"] \arrow[d, "\mathsf{op}_w^{\mathsf{Cell}}"'] & \mathsf{Cell}_n\, D \arrow[d, "\mathsf{op}_w^{\mathsf{Cell}}"] \\
\mathsf{Cell}_n\, \mathsf{op}_w\, C \arrow[r, "\mathsf{Cell}_n\, \mathsf{op}_w\, \sigma"'] & \mathsf{Cell}_n\, \mathsf{op}_w\, D
\end{tikzcd}$$

and that for all morphism $\tau \colon E \to C$,

$$\mathsf{op}_w\, \sigma \circ \mathsf{op}_w\, \tau = \mathsf{op}_w(\sigma \circ \tau).$$

By definition of $\mathsf{op}_w\, \sigma$, the square above commutes when restricted to generators. Moreover, for a coherence cell $c = \mathsf{coh}(B, A, \tau)$, we see that

$$\begin{aligned}
\mathsf{op}_w^{\mathsf{Cell}} \circ \mathsf{Cell}_n(\sigma)(c) &= \mathsf{op}_w^{\mathsf{Cell}}(\mathsf{coh}(B, A, \sigma \circ \tau)) \\
&= \mathsf{coh}(B, A', \mathsf{op}_w(\sigma \circ \tau) \circ \mathsf{Free}(\mathsf{op}_w^B)) \\
&= \mathsf{coh}(B, A', \mathsf{op}_w(\sigma) \circ \mathsf{op}_w(\tau) \circ \mathsf{Free}(\mathsf{op}_w^B)) \\
&= \mathsf{Cell}_n(\mathsf{op}_w\, \sigma)(\mathsf{coh}(B, A', \mathsf{op}_w(\tau) \circ \mathsf{Free}(\mathsf{op}_w^B))) \\
&= \mathsf{Cell}_n(\mathsf{op}_w\, \sigma) \circ \mathsf{op}_w^{\mathsf{Cell}}(\mathsf{coh}(B, A, \tau))
\end{aligned}$$

where A' is defined as above. Given arbitrary $\tau \colon E \to C$, we may assume that the square commutes when restricted to the image of τ_V. By the inductive hypothesis, op_w preserves composition of morphisms of $(n-1)$-computads. Hence it suffices to show the equality above for the generators of E. We recall the definition of the composition of morphisms of n-computads given in [11, Section 3.1]:

$$(\sigma_{n-1}, \sigma_v) \circ (\tau_{n-1}, \tau_V) = (\sigma_{n-1} \circ \tau_{n-1}, \mathsf{Cell}_n(\sigma) \circ \tau_v)$$

Using this definition, we have:

$$\begin{aligned}
(\mathsf{op}_w(\sigma \circ \tau))_V &= \mathsf{op}_w^{\mathsf{Cell}} \circ \mathsf{Cell}_n(\sigma) \circ \tau_V \\
&= \mathsf{Cell}_n(\mathsf{op}_w\, \sigma) \circ \mathsf{op}_n^{\mathsf{Cell}} \circ \tau_V \\
&= (\mathsf{op}_w(\sigma) \circ \mathsf{op}_w(\tau))_V.
\end{aligned}$$

Therefore, op_w is a functor and $\mathsf{op}_w^{\mathsf{Cell}}$ is natural.

Spheres. The natural transformation $\mathsf{op}_w^{\mathsf{Sphere}}$ is completely determined by Property (OP3). Indeed, for an n-computad C and for a sphere $(a,b) \in \mathsf{Sphere}_n\, C$, we are forced to define

$$\mathsf{op}_w^{\mathsf{Sphere}}(a,b) = \begin{cases} (\mathsf{op}_w^{\mathsf{Cell}}\, b, \mathsf{op}_w^{\mathsf{Cell}}\, a), & \text{if } n+1 \in w \\ (\mathsf{op}_w^{\mathsf{Cell}}\, a, \mathsf{op}_w^{\mathsf{Cell}}\, b), & \text{if } n+1 \notin w \end{cases}$$

Property (OP2) shows us that those $\mathsf{op}_w^{\mathsf{Cell}}\, a$ and $\mathsf{op}_w^{\mathsf{Cell}}\, b$ have the same source and target, so that this assignment is well-defined. It is clearly natural by naturality of $\mathsf{op}_w^{\mathsf{Cell}}$.

Fullness. To finish the recursive definition, it remains to show that for every Batanin tree B and every n-sphere $A = (a,b)$ of $\mathsf{Free}_n\, \mathsf{Pos}(B)$, the n-sphere

$$A' = \mathsf{Sphere}_n\, \mathsf{Free}_n(\mathsf{op}_w^B)^{-1}(\mathsf{op}_w^{\mathsf{Sphere}}(A))$$

of $\mathsf{Free}_n\, \mathsf{Pos}(\mathsf{op}_w\, B)$ is also full. We will show that in the case that $n+1 \in w$, the other case being similar. By assumption, we may write

$$a = \mathsf{Cell}_n\, \mathsf{Free}_n(s_n^B)(a_0) \qquad\qquad b = \mathsf{Cell}_n\, \mathsf{Free}_n(t_n^B)(b_0).$$

For n-cells a_0, b_0 of $\mathsf{Free}_n\, \mathsf{Pos}(\partial_n B)$ whose support contains all positions of $\partial_n B$. Then we have that $A' = (a', b')$ where

$$\begin{aligned} a' &= \mathsf{Cell}_n\, \mathsf{Free}_n((\mathsf{op}_w^B)^{-1} \circ t_n^B)(\mathsf{op}_w^{\mathsf{Cell}}\, b_0) \\ b' &= \mathsf{Cell}_n\, \mathsf{Free}_n((\mathsf{op}_w^B)^{-1} \circ t_n^B)(\mathsf{op}_w^{\mathsf{Cell}}\, a_0) \end{aligned}$$

By Lemma 7, we can rewrite those cells as

$$\begin{aligned} a' &= \mathsf{Cell}_n\, \mathsf{Free}_n(s_n^{\mathsf{op}_w B})(\mathsf{Cell}_n\, \mathsf{Free}_n(\mathsf{op}_w^{\partial_n B})^{-1}(\mathsf{op}_w^{\mathsf{Cell}}\, b_0)) \\ b' &= \mathsf{Cell}_n\, \mathsf{Free}_n(t_n^{\mathsf{op}_w B})(\mathsf{Cell}_n\, \mathsf{Free}_n(\mathsf{op}_w^{\partial_n B})^{-1}(\mathsf{op}_w^{\mathsf{Cell}}\, a_0)). \end{aligned}$$

Using the definition of the support and that $\mathsf{op}_w^{\mathsf{Cell}}$ preserves generators, we may show recursively that

$$\mathsf{supp}(\mathsf{op}_w^{\mathsf{Cell}}(c)) = \mathsf{supp}(c)$$

for every cell c. Moreover, isomorphisms of computads induce bijections on the support of cells, so the support of the cells

$$\mathsf{Cell}_n\, \mathsf{Free}_n(\mathsf{op}_w^{\partial_n B})^{-1}(\mathsf{op}_w^{\mathsf{Cell}}\, b_0)$$
$$\mathsf{Cell}_n\, \mathsf{Free}_n(\mathsf{op}_w^{\partial_n B})^{-1}(\mathsf{op}_w^{\mathsf{Cell}}\, a_0)$$

must contain all positions of $\partial_n\, \mathsf{op}_w\, B$. Therefore, A' is full.

Lemma 8. *For every $n \in \mathbb{N}$, the endofunctor $\mathsf{op}_\emptyset$ on n-computads is the identity, and so are the natural transformations $\mathsf{op}_\emptyset^{\mathsf{Cell}}$ and $\mathsf{op}_\emptyset^{\mathsf{Sphere}}$. Moreover, for any pair of elements $w, w' \in G$,*

$$\mathsf{op}_w\, \mathsf{op}'_w = \mathsf{op}_{ww'}$$

and the following diagrams commute.

$$
\begin{tikzcd}
\mathsf{Cell}_n \arrow[r, Rightarrow, "\mathsf{op}^{\mathsf{Cell}}_{w'}"] \arrow[d, equal] & \mathsf{Cell}_n \, \mathsf{op}_{w'} \arrow[r, Rightarrow, "\mathsf{op}^{\mathsf{Cell}}_{w} \, \mathsf{op}_{w'}"] & \mathsf{Cell}_n \, \mathsf{op}_w \, \mathsf{op}_{w'} \arrow[d, equal] \\
\mathsf{Cell}_n \arrow[rr, Rightarrow, "\mathsf{op}^{\mathsf{Cell}}_{ww'}"] & & \mathsf{Cell}_n \, \mathsf{op}_{ww'}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathsf{Sphere}_n \arrow[r, Rightarrow, "\mathsf{op}^{\mathsf{Sphere}}_{w'}"] \arrow[d, equal] & \mathsf{Cell}_n \, \mathsf{op}_{w'} \arrow[r, Rightarrow, "\mathsf{op}^{\mathsf{Sphere}}_{w} \, \mathsf{op}_{w'}"] & \mathsf{Sphere}_n \, \mathsf{op}_w \, \mathsf{op}_{w'} \arrow[d, equal] \\
\mathsf{Sphere}_n \arrow[rr, Rightarrow, "\mathsf{op}^{\mathsf{Sphere}}_{ww'}"] & & \mathsf{Sphere}_n \, \mathsf{op}_{ww'}
\end{tikzcd}
$$

In particular, op_w, $\mathsf{op}^{\mathsf{Cell}}_w$ *and* $\mathsf{op}^{\mathsf{Sphere}}_w$ *are invertible with inverses* op_w, $\mathsf{op}^{\mathsf{Cell}}_w \, \mathsf{op}_w$ *and* $\mathsf{op}^{\mathsf{Sphere}}_w \, \mathsf{op}_w$ *respectively.*

Proof. We proceed inductively on $n \in \mathbb{N}$, since the result holds trivially for $n = -1$. Since $\mathsf{op}_\emptyset$ and $\mathsf{op}^{\mathsf{Sphere}}_\emptyset$ are identities for $(n-1)$-computads, we see that

$$\mathsf{op}_\emptyset \, C = C$$

for every n-comptutad C. Using Lemma 7, we can then show mutually recursively for an n-computad C that

$$\mathsf{op}_\emptyset \, \sigma = \sigma \qquad\qquad \mathsf{op}^{\mathsf{Cell}}_\emptyset \, c = c$$

for every morphism $\sigma \colon D \to C$ and every n-cell c of C. Using then that $\mathsf{op}^{\mathsf{Sphere}}_\emptyset$ is defined using $\mathsf{op}^{\mathsf{Cell}}_\emptyset$, we see that $\mathsf{op}^{\mathsf{Sphere}}_\emptyset$ must be the identity as well.

Let now $w, w' \in G$ and $C = (C_{n-1}, V^C_n, \phi^C_n)$ an n-computad. Then $\mathsf{op}_w \, \mathsf{op}'_w \, C$ consists of the $(n-1)$-computad

$$\mathsf{op}_w \, \mathsf{op}'_w \, C_{n-1} = \mathsf{op}_{ww'} \, C_{n-1},$$

the same set of generators, and the attaching function

$$\phi^{\mathsf{op}_w \, \mathsf{op}'_w \, C}_n = \mathsf{op}^{\mathsf{Sphere}}_{w, \mathsf{op}_{w'} \, C} \circ \mathsf{op}^{\mathsf{Sphere}}_{w', C} \circ \phi^C_n = \mathsf{op}^{\mathsf{Sphere}}_{ww', C} \circ \phi^C_n = \phi^{\mathsf{op}_{ww'} \, C}_n.$$

Hence, $\mathsf{op}_w \, \mathsf{op}_{w'}$ and $\mathsf{op}_{ww'}$ agree on n-computads. Fixing a computad C, we can show that they also agree on morphisms with target C mutually inductively to recursively to showing that the claimed diagram for $\mathsf{op}^{\mathsf{Cell}}_w$ commutes. The commutative diagram from $\mathsf{op}^{\mathsf{Sphere}}_w$ then follows from the one for $\mathsf{op}^{\mathsf{Cell}}_w$. □

Having defined the opposite of an n-computad for every $n \in \mathbb{N}$, in a way that is compatible with the forgetful functors u_n, we get a functor

$$\mathsf{op}_w \colon \mathsf{Comp} \to \mathsf{Comp}$$

sending a computad $C = (C_n)_{n \in \mathbb{N}}$ to the computad

$$\mathsf{op}_w \, C = (\mathsf{op}_w \, C_n)_{n \in \mathbb{N}}$$

and acts similarly on morphisms. Property (OP1) shows that op_w is compatible with the inclusion functors Free in that

$$\begin{tikzcd} \mathsf{Comp} \arrow[r, "\mathsf{op}_w"] & \mathsf{Comp} \\ \mathsf{Glob} \arrow[u, "\mathsf{Free}"] \arrow[r, "\mathsf{op}_w"] & \mathsf{Glob} \arrow[u, "\mathsf{Free}"'] \end{tikzcd}$$

commutes. Moreover, combining Properties (OP2) and (OP3), we see that the natural transformations $\mathsf{op}_w^{\mathsf{Cell}}$ give rise to a natural transformation

$$\mathsf{op}_w^{\mathsf{Cell}} \colon \mathsf{op}_w \,\mathsf{Cell} \Rightarrow \mathsf{Cell}\, \mathsf{op}_w$$

The following lemma is an easy consequence of Lemma 8.

Lemma 9. *The functor* $\mathsf{op}_\emptyset \colon \mathsf{Comp} \to \mathsf{Comp}$ *is the identity functor, and the natural transformations* $\mathsf{op}_\emptyset^{\mathsf{Cell}}$ *is the identity of* Cell*. Moreover, for any pair* $w, w' \in G$*,*

$$\mathsf{op}_w \,\mathsf{op}'_w = \mathsf{op}_{ww'}$$

and the following diagrams commute.

$$\begin{tikzcd} \mathsf{op}_w \,\mathsf{op}_{w'} \,\mathsf{Cell} \arrow[r, Rightarrow, "\mathsf{op}_w \,\mathsf{op}_{w'}^{\mathsf{Cell}}"] \arrow[d, equal] & \mathsf{op}_w \,\mathsf{Cell}\, \mathsf{op}_{w'} \arrow[r, Rightarrow, "\mathsf{op}_w^{\mathsf{Cell}} \,\mathsf{op}_{w'}"] & \mathsf{Cell}\, \mathsf{op}_w \,\mathsf{op}_{w'} \arrow[d, equal] \\ \mathsf{op}_{ww'} \,\mathsf{Cell} \arrow[rr, Rightarrow, "\mathsf{op}_{ww'}^{\mathsf{Cell}}"'] & & \mathsf{Cell}\, \mathsf{op}_{ww'}. \end{tikzcd}$$

In particular, each op_w *is invertible with inverse itself, and* $\mathsf{op}_w^{\mathsf{Cell}}$ *is invertible with inverse* $\mathsf{op}_w \,\mathsf{op}_w^{\mathsf{Cell}} \,\mathsf{op}_w$*.*

4.3 The opposite of an ω-category.

So far, we have defined the opposite of a globular set, a pasting diagram and a computad. To extend those definitions and define the opposite of an ω-category, we observe first that the functor $\mathsf{op}_w : \mathsf{Glob} \to \mathsf{Glob}$ together with the natural transformation

$$\mathsf{op}_w^T : T \,\mathsf{op}_w \Rightarrow \mathsf{op}_w \,T$$
$$\mathsf{op}_w^T = (\mathsf{op}_w^{\mathsf{Cell}} \,\mathsf{Free})^{-1}$$

is a morphism of monads from T to T [1]. This amounts to commutativity of the following two diagrams

$$\begin{tikzcd} T\,\mathsf{op}_w \arrow[rr, Rightarrow, "\mathsf{op}_w^T"] & & \mathsf{op}_w \,T \\ & \mathsf{op}_w \arrow[ul, Rightarrow, "\mathsf{op}_w \,\eta"] \arrow[ur, Rightarrow, "\eta \,\mathsf{op}_w"'] & \end{tikzcd} \qquad \begin{tikzcd} TT\,\mathsf{op}_w \arrow[r, Rightarrow, "T\,\mathsf{op}_w^T"] \arrow[d, Rightarrow, "\mu \,\mathsf{op}_w"'] & T\,\mathsf{op}_w \,T \arrow[r, Rightarrow, "\mathsf{op}_w^T \,T"] & \mathsf{op}_w \,TT \arrow[d, Rightarrow, "\mathsf{op}_w \,\mu"] \\ T\,\mathsf{op}_w \arrow[rr, Rightarrow, "\mathsf{op}_w^T"] & & \mathsf{op}_w \,T \end{tikzcd}$$

[1] op_w^T can also be obtained as the mate of $\mathsf{op}_w^{\mathsf{Cell}} \,\mathsf{Free}$ under the $\mathsf{op}_w \dashv \mathsf{op}_w$ adjunction.

The left one is the assertion that $\mathsf{op}_w^{\mathsf{Cell}}$ preserves generators, which we have already shown. The right one is obtained from the following diagram by whiskering on the right with Free, and then replacing $\mathsf{op}^{\mathsf{Cell}}\,\mathsf{Free}$ with its inverse.

$$\begin{tikzcd}
\mathsf{op}_w\, T\, \mathsf{Cell} \arrow[r, Rightarrow, "\mathsf{op}_w^{\mathsf{Cell}}\,\mathsf{Free}\,\mathsf{Cell}"] \arrow[d, Rightarrow, "\mathsf{op}_w\,\mathsf{Cell}\,\varepsilon"'] & \mathsf{Cell}\,\mathsf{op}_w\,\mathsf{Free}\,\mathsf{Cell} \arrow[r, equal] & T\,\mathsf{op}_w\,\mathsf{Cell} \arrow[r, Rightarrow, "T\,\mathsf{op}_w^{\mathsf{Cell}}"] & T\,\mathsf{Cell}\,\mathsf{op}_w \arrow[d, Rightarrow, "\mathsf{Cell}\,\varepsilon\,\mathsf{op}_w"] \\
\mathsf{op}_w\,\mathsf{Cell} \arrow[rrr, Rightarrow, "\mathsf{op}_w^{\mathsf{Cell}}"'] & & & \mathsf{Cell}\,\mathsf{op}_w
\end{tikzcd}$$

To show that this diagram commutes, we fix a computad C and proceed inductively on the cells of $\mathsf{Free}\,\mathsf{Cell}\,C$. Given a generator $v \in V_n^{\mathsf{Free}\,\mathsf{Cell}\,C}$, we compute that

$$\begin{aligned}
((\mathsf{Cell}\,\varepsilon_{\mathsf{op}_w\,C}) \circ (T\,\mathsf{op}_{w,C}^{\mathsf{Cell}}) \circ (\mathsf{op}_{w,\mathsf{Free}\,\mathsf{Cell}\,C}^{\mathsf{Cell}}))(\mathsf{var}\,c)& \\
&= ((\mathsf{Cell}\,\varepsilon_{\mathsf{op}_w\,C}) \circ (T\,\mathsf{op}_{w,C}^{\mathsf{Cell}}))(\mathsf{var}\,c) \\
&= (\mathsf{Cell}\,\varepsilon_{\mathsf{op}_w\,C})(\mathsf{var}\,\mathsf{op}_{w,C}^{\mathsf{Cell}}\,c) \\
&= \mathsf{op}_{w,C}^{\mathsf{Cell}}(c) \\
&= \mathsf{op}_{w,C}^{\mathsf{Cell}}((\mathsf{op}_w\,\mathsf{Cell}\,\varepsilon_C)(c))
\end{aligned}$$

so the diagram commutes when restricted to generators. Let now $c = \mathsf{coh}(B, A, \tau)$ a coherence cell of $\mathsf{Free}\,\mathsf{Cell}\,C$. Then letting

$$A' = \mathsf{Sphere}_{n-1}\,\mathsf{Free}_{n-1}(\mathsf{op}_w^B)^{-1}(\mathsf{op}_w^{\mathsf{Sphere}}(A)),$$

we compute that

$$\begin{aligned}
&((\mathsf{Cell}\,\varepsilon_{\mathsf{op}_w\,C}) \circ (T\,\mathsf{op}_{w,C}^{\mathsf{Cell}}) \circ (\mathsf{op}_{w,\mathsf{Free}\,\mathsf{Cell}\,C}^{\mathsf{Cell}}))(\mathsf{coh}(B, A, \tau)) \\
&\qquad = \mathsf{coh}(\mathsf{op}_w\,B, A', \varepsilon_{\mathsf{op}_w\,C} \circ \mathsf{Free}_n(\mathsf{op}_{w,C}^{\mathsf{Cell}}) \circ \mathsf{op}_w\,\tau \circ \mathsf{Free}_n(\mathsf{op}_w^B)) \\
&\mathsf{op}_{w,C}^{\mathsf{Cell}}((\mathsf{op}_w\,\mathsf{Cell}\,\varepsilon_C)(c)) \\
&\qquad = \mathsf{coh}(\mathsf{op}_w\,B, A', \mathsf{op}_w(\varepsilon_C \circ \tau) \circ \mathsf{Free}_n(\mathsf{op}_w^B)),
\end{aligned}$$

so it suffices to show that the following equality of morphisms:

$$\mathsf{op}_w(\varepsilon_C \circ \tau) = \varepsilon_{\mathsf{op}_w\,C} \circ \mathsf{Free}_n(\mathsf{op}_{w,C}^{\mathsf{Cell}}) \circ \mathsf{op}_w\,\tau$$

By induction on the dimension and the structure of cells, we may assume that the diagram commutes when restricted to cells of the form $\tau_{k,V}(v)$ for $k \le n$ and $v \in \mathsf{Pos}_k(B)$. Then for every such v,

$$\begin{aligned}
(\mathsf{op}_w(\varepsilon_C \circ \tau))_{k,V}(v) &= \mathsf{op}_{w,C}^{\mathsf{Cell}}((\varepsilon_C \circ \tau)_{k,V}(v)) \\
&= (\mathsf{op}_{w,C}^{\mathsf{Cell}} \circ \mathsf{Cell}(\varepsilon_C))(\tau_{k,V}(v)) \\
&= (\mathsf{op}_{w,C}^{\mathsf{Cell}} \circ (\mathsf{op}_w\,\mathsf{Cell}(\varepsilon_C)))(\tau_{k,V}(v)) \\
&= ((\mathsf{Cell}\,\varepsilon_{\mathsf{op}_w\,C}) \circ (T\,\mathsf{op}_{w,C}^{\mathsf{Cell}}) \circ \mathsf{op}_{w,\mathsf{Free}\,\mathsf{Cell}\,C}^{\mathsf{Cell}})(\tau_{k,V}(v)) \\
&= \mathsf{Cell}(\varepsilon_{\mathsf{op}_w\,C} \circ \mathsf{Free}\,\mathsf{op}_{w,C}^{\mathsf{Cell}})((\mathsf{op}_w\,\tau)_{k,V}(v)) \\
&= (\varepsilon_{\mathsf{op}_w\,C} \circ \mathsf{Free}\,\mathsf{op}_{w,C}^{\mathsf{Cell}} \circ \mathsf{op}_w\,\tau)_{k,V}(v),
\end{aligned}$$

so the morphisms agree on every generator of $\mathsf{Free}\,\mathsf{Pos}(B)$. Thus, they must be equal, and the square commutes for coherence cells as well.

Definition 10. The opposite of an ω-category $(X, \alpha : TX \to X)$ with respect to some $w \in G$ is the ω-category consisting of the globular set $\mathsf{op}_w X$ and the structure morphism

$$T\,\mathsf{op}_w X \xrightarrow{\mathsf{op}^T_{w,X}} \mathsf{op}_w TX \xrightarrow{\mathsf{op}_w\,\alpha} \mathsf{op}_w X$$

The construction of the opposite of an ω-category is well-defined and gives rise to endofunctors

$$\mathsf{op}_w : \omega\,\mathsf{Cat} \to \omega\,\mathsf{Cat}$$

for every $w \in G$ as shown by Street [21], and explained by Leinster [17, Theorem 6.1.1]. Moreover, the following lemma - an immediate consequence of Lemma 9 - shows that those endofunctors are invertible and give rise to an action

$$\mathsf{op} : G \to \mathsf{Aut}(\omega\,\mathsf{Cat})$$

of G on the category of ω-categories.

Lemma 11. *The natural transformation $\mathsf{op}^T_\emptyset$ is the identity, and for any $w, w' \in G$ the following diagram commutes:*

$$\begin{tikzcd}
T\,\mathsf{op}_w\,\mathsf{op}_{w'} \arrow[r, Rightarrow, "\mathsf{op}^T_w\,\mathsf{op}_{w'}"] \arrow[d, equal] & \mathsf{op}_w\,T\,\mathsf{op}_{w'} \arrow[r, Rightarrow, "\mathsf{op}_w\,\mathsf{op}^T_{w'}"] & \mathsf{op}_w\,\mathsf{op}_{w'}\,T \arrow[d, equal] \\
T\,\mathsf{op}_{ww'} \arrow[rr, Rightarrow, "\mathsf{op}^T_{ww'}"'] & & \mathsf{op}_{ww'}\,T
\end{tikzcd}$$

Since ω-categories are algebras for the monad (T, μ, η) on Glob induced by the adjunction $\mathsf{Free} \dashv \mathsf{Cell}$, there exists a free/underlying adjunction

$$F^T : \mathsf{Glob} \rightleftarrows \omega\,\mathsf{Cat} : U^T$$

between globular sets and ω-categories, and there exists a comparison functor

$$K^T : \mathsf{Comp} \to \omega\,\mathsf{Cat}$$

sending a computad C to the ω-category $(\mathsf{Cell}\,C, \mathsf{Cell}\,\varepsilon_C)$. Moreover, K^T is a morphism of adjunctions meaning that

$$F^T = K^T\,\mathsf{Free} \qquad\qquad \mathsf{Cell} = U^T K^T$$

The commutative diagram showing that op^T_w is a morphism of monads implies in particular that the components of the natural transformation $\mathsf{op}^{\mathsf{Cell}}_w$ are morphisms of free ω-categories, so it can also be seen as a natural isomorphism

$$\mathsf{op}^K_w : \mathsf{op}_w K^T \Rightarrow K^T\,\mathsf{op}_w$$

The opposite functors on globular sets, computads and ω-categories are therefore related by the following five squares

$$
\begin{tikzcd}
\omega\,\mathsf{Cat} \arrow[r, "\mathsf{op}_w"] \arrow[d, "U^T"'] & \omega\,\mathsf{Cat} \arrow[d, "U^T"] \\
\mathsf{Glob} \arrow[r, "\mathsf{op}_w"] & \mathsf{Glob}
\end{tikzcd}
\qquad
\begin{tikzcd}
\omega\,\mathsf{Cat} \arrow[r, "\mathsf{op}_w"] & \omega\,\mathsf{Cat} \\
\mathsf{Comp} \arrow[r, "\mathsf{op}_w"] \arrow[u, "K^T"] \arrow[ur, Rightarrow, "\mathsf{op}_w^K"] & \mathsf{Comp} \arrow[u, "K^T"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Comp} \arrow[r, "\mathsf{op}_w"] & \mathsf{Comp} \\
\mathsf{Glob} \arrow[r, "\mathsf{op}_w"] \arrow[u, "\mathsf{Free}"] & \mathsf{Glob} \arrow[u, "\mathsf{Free}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathsf{Comp} \arrow[r, "\mathsf{op}_w"] \arrow[d, "\mathsf{Cell}"'] & \mathsf{Comp} \arrow[d, "\mathsf{Cell}"] \\
\mathsf{Glob} \arrow[r, "\mathsf{op}_w"] \arrow[ur, Rightarrow, "\mathsf{op}_w^{\mathsf{Cell}}"] & \mathsf{Glob}
\end{tikzcd}
\qquad
\begin{tikzcd}
\omega\,\mathsf{Cat} \arrow[r, "\mathsf{op}_w"] & \omega\,\mathsf{Cat} \\
\mathsf{Glob} \arrow[r, "\mathsf{op}_w"] \arrow[u, "F^T"] \arrow[ur, Rightarrow, "\mathsf{op}_w^F"] & \mathsf{Glob} \arrow[u, "F^T"']
\end{tikzcd}
$$

where

$$\mathsf{op}_w^F = \mathsf{op}_w^K\,\mathsf{Free} \qquad\qquad \mathsf{op}_w^{\mathsf{Cell}} = U^T\,\mathsf{op}_w^K\,.$$

In particular, the opposite of an ω-category that is free on a globular set or computad is again free on the opposite of the underlying globular set or the opposite computad, up to natural isomorphism.

5 The suspension and hom functors

Strict ω-categories are precisely categories enriched over strict ω-categories, so for every strict ω-category X and every pair of 0-cells $x_-, x_+ \in X_0$, the globular set $\Omega(X, x,x_+)$ of cells from x_- to x_+ admits an ω-category structure in a functorial way [9]. The same result was recently proven for arbitrary ω-categories [10] using the operadic definition of Leinster. To illustrate the power of the inductive techniques used in the previous section, we will provide an alternative, elementary proof of this result, and show that the formation of opposites and hom ω-categories commute in a suitable sense.

Recall that the *path space* or *hom* functor $\Omega : \mathsf{Glob}^{\star,\star} \to \mathsf{Glob}$, taking a bipointed globular set to the globular set of cells from the first basepoint to the second one admits a left adjoint, the *suspension* functor, $\Sigma : \mathsf{Glob} \to \mathsf{Glob}^{\star,\star}$ with unit the identity and counit $\kappa : \Sigma\,\Omega \Rightarrow \mathsf{id}$ given by subset inclusions. Our goal in this section will be to extend the suspension functor to computads, and the path space functor on ω-categories.

To do so, we let first $\mathsf{Comp}^{\star,\star}$ the category of computads with two chosen 0-cells and morphisms preserving those 0-cells. By the $\mathsf{Free} \dashv \mathsf{Cell}$ adjunction and the Yoneda lemma, this is precisely the slice of Comp under $\mathsf{Free}(\mathbb{D}^0 + \mathbb{D}^0)$. The adjunction descends to the slices to give an adjunction

$$\mathsf{Free}^{\star,\star} : \mathsf{Glob}^{\star,\star} \rightleftarrows \mathsf{Comp}^{\star,\star} : \mathsf{Cell}^{\star,\star}$$

where $\mathsf{Free}^{\star,\star}(X, x_-, x_+)$ is the computad $\mathsf{Free}\,X$ with the 0-cells $\mathsf{var}\,x_-$ and $\mathsf{var}\,x_+$, and $\mathsf{Cell}^{\star,\star}(C, c_-, c_+)$ is the globular set $\mathsf{Cell}\,C$ with the basepoints c_- and c_+ respectively. The unit and counit of this adjunction are given by those of the original adjunction seen as pointed map. The monad $T^{\star,\star} : \mathsf{Glob}^{\star,\star} \to \mathsf{Glob}^{\star,\star}$ induced by this adjunction sends a bipointed globular set (X, x_-, x_+) to the globular set TX with basepoints $\mathsf{var}\, x_-$ and $\mathsf{var}\, x_+$. Its category of algebras can be seen to be the category $\omega\,\mathsf{Cat}^{\star,\star}$ of ω-categories with two chosen basepoints. Therefore, to define a hom ω-category functor

$$\Omega : \omega\,\mathsf{Cat}^{\star,\star} \to \omega\,\mathsf{Cat}$$

extending the hom functor $\Omega : \mathsf{Glob}^{\star,\star} \to \mathsf{Glob}$, in that the following square commutes

$$\begin{tikzcd}
\omega\,\mathsf{Cat}^{\star,\star} \arrow[r, "\Omega"] \arrow[d, "U^{T^{\star,\star}}"'] & \omega\,\mathsf{Cat} \arrow[d, "U^T"] \\
\mathsf{Glob}^{\star,\star} \arrow[r, "\Omega"] & \mathsf{Glob},
\end{tikzcd}$$

it suffices to define a natural transformation

$$\Omega^T : T\,\Omega \Rightarrow \Omega\, T^{\star,\star}$$

such that the pair (Ω, Ω^T) is a morphism of monads from T to $T^{\star,\star}$. To define such a natural transformation, it suffices to define its mate [15, Proposition 2.1]

$$\Sigma^T : \Sigma\, T \Rightarrow T^{\star,\star}\, \Sigma$$

which we will define inductively together with an extension of the suspension functor to computads.

5.1 The suspension of a computad

We define first the *suspension* of a Batanin tree B to be the Batanin tree

$$\Sigma\, B = \mathsf{br}[B]$$

since by definition

$$\mathsf{Pos}^{\star,\star}(\Sigma\, B) = \Sigma\, \mathsf{Pos}(B).$$

We will then proceed inductively on $n \in \mathbb{N}$ to define a functor and two natural transformations

$$\Sigma : \mathsf{Comp}_n \to \mathsf{Comp}_{n+1}$$
$$\Sigma^{\mathsf{Cell}} : \mathsf{Cell}_n \Rightarrow \mathsf{Cell}_{n+1}\, \Sigma$$
$$\Sigma^{\mathsf{Sphere}} : \mathsf{Sphere}_n \Rightarrow \mathsf{Sphere}_{n+1}\, \Sigma$$

satisfying the following properties:

(S1) the suspension commutes with the forgetful functors, and the inclusion of globular sets into computads:

$$\begin{tikzcd}
\mathsf{Comp}_{n+1} \arrow[r, "\Sigma"] \arrow[d, "u_{n+1}"] & \mathsf{Comp}_{n+2} \arrow[d, "u_{n+2}"] \\
\mathsf{Comp}_n \arrow[r, "\Sigma"'] & \mathsf{Comp}_{n+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Comp}_n \arrow[r, "\Sigma"] & \mathsf{Comp}_{n+1} \\
\mathsf{Glob} \arrow[u, "\mathsf{Free}_n"] \arrow[r, "\Sigma"'] & \mathsf{Glob} \arrow[u, "\mathsf{Free}_{n+1}"']
\end{tikzcd}$$

(S2) the natural transformations are compatible with the boundary natural transformations:

$$\begin{tikzcd} \mathsf{Cell}_{n+1} \arrow[rr, Rightarrow, "\Sigma^{\mathsf{Cell}}"] \arrow[d, Rightarrow, "\mathsf{bdry}_{n+1}"'] & & \mathsf{Cell}_{n+2}\,\Sigma \arrow[d, Rightarrow, "\mathsf{bdry}_{n+1}\,\Sigma"] \\ \mathsf{Sphere}_n\, u_{n+1} \arrow[r, Rightarrow, "\Sigma^{\mathsf{Sphere}}\, u_{n+1}"'] & \mathsf{Sphere}_n\,\Sigma\, u_{n+1} \arrow[r, equal] & \mathsf{Sphere}_n\, u_{n+1}\,\Sigma \end{tikzcd}$$

(S3) the natural transformations are compatible with the projection natural transformations for $i = 1, 2$:

$$\begin{tikzcd} \mathsf{Sphere}_n \arrow[r, Rightarrow, "\Sigma^{\mathsf{Sphere}}"] \arrow[d, Rightarrow, "\mathsf{pr}_i"] & \mathsf{Sphere}_n \arrow[d, Rightarrow, "\mathsf{pr}_i"] \\ \mathsf{Cell}_n \arrow[r, Rightarrow, "\Sigma^{\mathsf{Cell}}"'] & \mathsf{Cell}_n \end{tikzcd}$$

(S4) the natural transformation Σ^{Cell} preserves generators, in the for every globular set X and $x \in X_n$, we have that

$$\Sigma^{\mathsf{Cell}}(\mathsf{var}\, x) = \mathsf{var}\, x$$

(S5) the natural transformation Σ^{Sphere} preserves fullness, in that for every full n-sphere A of $\mathsf{Free}\,\mathsf{Pos}(B)$, we have that $\Sigma^{\mathsf{Sphere}} A$ is a full $(n+1)$-sphere of $\mathsf{Free}\,\mathsf{Pos}(\Sigma\, B)$.

To start the induction, we recall that Comp_{-1} is the terminal category and that Comp_0 is the category Set of sets. The suspension functor is defined as the functor picking the 2-element set $\{v_-, v_+\}$. We recall also that the unique (-1)-computad has a unique (-1)-sphere. We define Σ^{Sphere} to be the natural transformation picking the 0-sphere (v_-, v_+). This concludes the base case. We will now assume that we have defined the the data satisfying the properties we have cited, up to dimension $n-1$, for a fixed $n \in \mathbb{N}$.

Computads. We will first define the functor Σ on all objects: Given an n-computad $C = (C_{n-1}, V_n^C, \phi_n^C)$, its suspension is the $(n+1)$-computad consisting of $\Sigma\, C_{n-1}$, the same set of generators, and the attaching function $\phi_{n+1}^{\Sigma\, C}$ given by the composite

$$\phi_{n+1}^{\Sigma\, C} : V_n^C \xrightarrow{\phi_n^C} \mathsf{Sphere}_{n-1}\, C_{n-1} \xrightarrow{\Sigma^{\mathsf{Sphere}}} \mathsf{Sphere}_n\, \Sigma\, C_{n-1}$$

By definition, the suspension functor on objects commutes with the forgetful functors. Using that the suspension functor on $(n-1)$-computads commutes also with the inclusion of globular sets, and that Σ^{Cell} and hence Σ^{Sphere} preserve generators, we see that the suspension functor on n-computads also commutes with the inclusions.

Cells and morphisms. We then define the suspension of a morphism of n-computads together with the natural transformation Σ^{Cell} mutually inductively, while showing that Property (S2) holds. Given a morphism of n-computads σ of target $C = (C_{n-1}, V_n^C, \phi_n^C)$, we define Σ_C^{Cell} and $\Sigma(\sigma)$. For a generator $v \in V_n^C$, we let

$$\Sigma_C^{\mathsf{Cell}}(\mathsf{var}\, v) = \mathsf{var}\, v,$$

and we compute that

$$(\mathsf{bdry}_{n+1,\Sigma C}\, \Sigma_C^{\mathsf{Cell}})(\mathsf{var}\, v) = \phi_{n+1}^{\Sigma C}(v) = (\Sigma_{C_{n-1}}^{\mathsf{Sphere}}\, \mathsf{bdry}_{n,C})(\mathsf{var}\, v).$$

For a coherence n-cell $c = \mathsf{coh}(B, A, \tau)$ of C, we let

$$\Sigma_C^{\mathsf{Cell}}(\mathsf{coh}(B, A, \tau)) = \mathsf{coh}(\Sigma B, \Sigma^{\mathsf{Sphere}} A, \Sigma \tau)$$

using that the suspension commutes with Free_n and $\mathsf{Pos}(-)$, and that it preserves fullness. Then by naturality of Σ^{Sphere}, we compute that

$$(\mathsf{bdry}_{n+1,\Sigma C}\, \Sigma_C^{\mathsf{Cell}})(\mathsf{coh}(B, A, \tau)) = (\Sigma_{C_{n-1}}^{\mathsf{Sphere}}\, \mathsf{bdry}_{n,C})(\mathsf{coh}(B, A, \tau)).$$

For a morphism $\sigma : D \to C$, we let $\Sigma \sigma : \Sigma D \to \Sigma C$ consist of $\Sigma \sigma_{n-1}$ and the function

$$(\Sigma \sigma)_V : V_{n+1}^{\Sigma D} = V_n^D \xrightarrow{\sigma_V} \mathsf{Cell}_n(C) \xrightarrow{\Sigma_C^{\mathsf{Cell}}} \mathsf{Cell}_{n+1} \Sigma C$$

This is a well-defined morphism by the observation on the boundary of Σ^{Cell}. Functoriality of the suspension, and naturality of Σ^{Cell} can be shown mutually inductively in the same way that they were shown for the opposite functors.

Spheres. Finally, the natural transformation Σ^{Sphere} is defined for a computad C and an n-sphere $A = (a, b)$ of it again by

$$\Sigma^{\mathsf{Sphere}}(a, b) = (\Sigma^{\mathsf{Cell}}\, a, \Sigma^{\mathsf{Cell}}\, b).$$

We observe that those $(n+1)$-cells are parallel again by Property (S2).

Fullness. To finish the induction, it remains to show that for every Batanin tree B and full n-sphere $A = (a, b)$ of $\mathsf{Free}_n \mathsf{Pos}(B)$, the sphere $\Sigma^{\mathsf{Sphere}} A$ is full in $\mathsf{Free}_{n+1} \mathsf{Pos}(\Sigma B)$. To show that, we first let

$$a = \mathsf{Cell}_n\, \mathsf{Free}_n(s_n^B)(a_0) \qquad\qquad b = \mathsf{Cell}_n\, \mathsf{Free}_n(t_n^B)(b_0),$$

where the support of a_0 and b_0 contains all positions of $\partial_n B$. Then $\Sigma^{\mathsf{Sphere}} A$ consists of the cells

$$\begin{aligned} a' &= \Sigma^{\mathsf{Cell}}(\mathsf{Cell}_n\, \mathsf{Free}_n(s_n^B)(a_0)) \\ &= \mathsf{Cell}_n\, \Sigma\, \mathsf{Free}_n(s_n^B)(\Sigma^{\mathsf{Cell}}(a_0)) \\ &= \mathsf{Cell}_n\, \mathsf{Free}_n(\Sigma\, s_n^B)(\Sigma^{\mathsf{Cell}}(a_0)) \\ &= \mathsf{Cell}_n\, \mathsf{Free}_n(s_{n+1}^{\Sigma B})(\Sigma^{\mathsf{Cell}}(a_0)) \\ b' &= \mathsf{Cell}_n\, \mathsf{Free}_n(t_{n+1}^{\Sigma B})(\Sigma^{\mathsf{Cell}}(b_0)), \end{aligned}$$

so it remains to show that when the support of a contains all positions of a tree B, then the support of $\Sigma^{\mathsf{Cell}}\, a$ contains all positions of $\Sigma\, B$. More generally, it suffices to prove that for every computad C and every cell $c \in \mathsf{Cell}_n\, C$,

$$\mathsf{supp}(\Sigma^{\mathsf{Cell}}(c)) = \mathsf{supp}(c) \cup \{v_-, v_+\}.$$

This statement can be easily shown by structural induction on cells.

Infinite-dimensional computads. This concludes the induction on $n \in \mathbb{N}$. Compatibility of the suspension functors with the forgetful functors allows us to define a functor

$$\Sigma : \mathsf{Comp} \to \mathsf{Comp}^{\star,\star}$$

sending a computad $C = (C_n)_{n\in\mathbb{N}}$ to the computad with components

$$(\Sigma\, C)_0 = \{v_-, v_+\} \qquad\qquad (\Sigma\, C)_{n+1} = \Sigma\, C_n,$$

and with basepoints $\mathsf{var}\, v_-$ and $\mathsf{var}\, v_+$. By construction, this functor commutes with the suspension operation on globular sets, in that the following square commutes:

$$\begin{tikzcd} \mathsf{Comp} \arrow[r, "\Sigma"] & \mathsf{Comp}^{\star,\star} \\ \mathsf{Glob} \arrow[u, "\mathsf{Free}"] \arrow[r, "\Sigma"] & \mathsf{Glob}^{\star,\star} \arrow[u, "\mathsf{Free}^{\star,\star}"'] \end{tikzcd}$$

Moreover, Property (S2) shows that the natural transformations Σ^{Cell} can be combined to a natural transformation

$$\Sigma^{\mathsf{Cell}} : \Sigma\, \mathsf{Cell} \Rightarrow \mathsf{Cell}^{\star,\star}\, \Sigma\, .$$

Whiskering this natural transformation on the right with Free and using commutativity of the square above, we get the desired natural transformation

$$\Sigma^{T} : \Sigma\, T \Rightarrow T^{\star,\star}\, \Sigma\, .$$

5.2 Hom ω-categories

To get the path space functor, it remains to show that the mate

$$\Omega^{T} = (\Omega\, T^{\star,\star}\kappa) \circ (\Omega\, \Sigma^{T}\, \Omega) : T\, \Omega \Rightarrow \Omega\, T^{\star,\star}$$

is part of a morphism of monads, meaning that the following diagrams commute:

$$\begin{tikzcd} T\,\Omega \arrow[rr, Rightarrow, "\Omega^T"] & & \Omega\, T^{\star,\star} \\ & \Omega \arrow[ul, Rightarrow, "\eta\,\Omega"] \arrow[ur, Rightarrow, "\Omega\,\eta"'] & \end{tikzcd} \qquad \begin{tikzcd} T T\,\Omega \arrow[r, Rightarrow, "T\,\Omega^T"] \arrow[d, Rightarrow, "\mu\,\Omega"'] & T\,\Omega\, T^{\star,\star} \arrow[r, Rightarrow, "\Omega^T\, T^{\star,\star}"] & \Omega\, T^{\star,\star} T^{\star,\star} \arrow[d, Rightarrow, "\Omega\,\mu"] \\ T\,\Omega \arrow[rr, Rightarrow, "\Omega^T"'] & & \Omega\, T^{\star,\star} \end{tikzcd}$$

By the mate corresponce, commutativity of those diagrams is equivalent to the commutativity of the following ones:

$$\begin{tikzcd} \Sigma\,T \arrow[rr, Rightarrow, "\Sigma^T"] & & T^{\star,\star}\,\Sigma \\ & \Sigma \arrow[ul, Rightarrow, "\Sigma\,\eta"] \arrow[ur, Rightarrow, "\eta\,\Sigma"'] & \end{tikzcd} \qquad \begin{tikzcd} \Sigma\,TT \arrow[r, Rightarrow, "\Sigma^T\,T"] \arrow[d, Rightarrow, "\Sigma\,\mu"'] & T^{\star,\star}\,\Sigma\,T \arrow[r, Rightarrow, "T^{\star,\star}\,\Sigma^T"] & T^{\star,\star}T^{\star,\star}\,\Sigma \arrow[d, Rightarrow, "\mu\,\Sigma"] \\ \Sigma\,T \arrow[rr, Rightarrow, "\Sigma^T"'] & & T^{\star,\star}\,\Sigma \end{tikzcd}$$

The left one commutes, since Σ^{Cell} preserves generators. The right one is obtained from the following diagram by whiskering on the right with Free:

$$\begin{tikzcd} \Sigma\,T\,\mathsf{Cell} \arrow[r, Rightarrow, "\Sigma^{\mathsf{Cell}}\,\mathsf{Free}\,\mathsf{Cell}"] \arrow[d, Rightarrow, "\Sigma\,\mathsf{Cell}\,\varepsilon"'] & T^{\star,\star}\,\Sigma\,\mathsf{Cell} \arrow[r, Rightarrow, "T^{\star,\star}\,\Sigma^{\mathsf{Cell}}"] & T^{\star,\star}\,\mathsf{Cell}^{\star,\star}\,\Sigma \arrow[d, Rightarrow, "\mathsf{Cell}^{\star,\star}\,\varepsilon\,\Sigma"] \\ \Sigma\,\mathsf{Cell} \arrow[rr, Rightarrow, "\Sigma^{\mathsf{Cell}}"'] & & \mathsf{Cell}^{\star,\star}\,\Sigma \end{tikzcd}$$

To show that this diagram commutes, we fix a computad C and proceed inductively. The diagram commutes when restricted to 0-cells of $\Sigma\,T\,\mathsf{Cell}\,C$, since those are precisely the basepoints of the suspension. Let therefore $n \in \mathbb{N}$ and $c \in (\Sigma\,T\,\mathsf{Cell}\,C)_{n+1} = (T\,\mathsf{Cell}\,C)_n$ a cell. If $c = \mathsf{var}\,c'$ is a generator cell, then

$$(\mathsf{Cell}^{\star,\star}(\varepsilon_{\Sigma\,C}) \circ T^{\star,\star}(\Sigma_C^{\mathsf{Cell}}) \circ \Sigma_{\mathsf{Free}\,\mathsf{Cell}\,C}^{\mathsf{Cell}})(c) = \Sigma_C^{\mathsf{Cell}}(c') = (\Sigma_C^{\mathsf{Cell}} \circ \Sigma\,\mathsf{Cell}(\varepsilon_C))(c),$$

so the diagram commutes when restricted to generators. If $c = \mathsf{coh}(B, A, \tau)$ is a coherence cell, we may assume inductively that the diagram commutes when restricted to cells of the form $\tau_{k,V}(v)$ for $k \leq n$ and $v \in \mathsf{Pos}_k(B)$. Then we compute that

$$\begin{aligned} (\mathsf{Cell}^{\star,\star}(\varepsilon_{\Sigma\,C}) \circ T^{\star,\star}(\Sigma_C^{\mathsf{Cell}}) \circ \Sigma_{\mathsf{Free}\,\mathsf{Cell}\,C}^{\mathsf{Cell}})(c) & \\ &= \mathsf{coh}(\Sigma\,B, \Sigma^{\mathsf{Sphere}}\,A, \varepsilon_{\Sigma\,C} \circ \mathsf{Free}\,\Sigma_C^{\mathsf{Cell}} \circ \Sigma\,\tau) \\ (\Sigma_C^{\mathsf{Cell}} \circ \Sigma\,\mathsf{Cell}(\varepsilon_C))(c) & \\ &= \mathsf{coh}(\Sigma\,B, \Sigma^{\mathsf{Sphere}}\,A, \Sigma(\varepsilon_C \circ \tau)), \end{aligned}$$

so it remains to show that the following equality of morphisms holds.

$$\varepsilon_{\Sigma\,C} \circ \mathsf{Free}\,\Sigma_C^{\mathsf{Cell}} \circ \Sigma\,\tau = \Sigma(\varepsilon_C \circ \tau).$$

For that, we fix a position $v \in \mathsf{Pos}_k(B)$ and compute by the inductive hypothesis that

$$\begin{aligned} (\Sigma(\varepsilon_C \circ \tau))_V(v) &= \Sigma_C^{\mathsf{Cell}}((\varepsilon_C \circ \tau)_V(v)) \\ &= \Sigma^{\mathsf{Cell}}(\mathsf{Cell}(\varepsilon_C)(\tau_V(v))) \\ &= \mathsf{Cell}(\varepsilon_{\Sigma\,C} \circ \mathsf{Free}\,\Sigma_C^{\mathsf{Cell}})(\Sigma_{\mathsf{Free}\,\mathsf{Cell}\,C}^{\mathsf{Cell}}(\tau_V(v))) \\ &= \mathsf{Cell}(\varepsilon_{\Sigma\,C} \circ \mathsf{Free}\,\Sigma_C^{\mathsf{Cell}})((\Sigma\,\tau)_V(v)) \\ &= (\varepsilon_{\Sigma\,C} \circ \mathsf{Free}\,\Sigma_C^{\mathsf{Cell}} \circ \Sigma\,\tau)_V(v), \end{aligned}$$

so the morphisms are equal.

Definition 12. The *hom ω-category* of a bipointed ω-category $(X, \alpha : TX \to X, x_-, x_+)$ is the ω-category that consists of the globular set $\Omega(X, x, x_+)$ and the structure morphism

$$T\,\Omega(X, x, x_+) \xrightarrow{\Omega^T} \Omega\,T^{\star,\star} X \xrightarrow{\Omega\,\alpha} \Omega(X, x, x_+)$$

As explained above, this definition extends to a functor

$$\Omega : \omega\,\mathsf{Cat}^{\star,\star} \to \omega\,\mathsf{Cat}$$

that extends the path space functor $\Omega : \mathsf{Glob}^{\star,\star} \to \mathsf{Glob}$. Given a bipointed ω-category X, the structure morphism $\alpha_{\Omega\,X} : T\,\Omega\,X \to \Omega\,X$ can be described as follows: for every cell $x \in (\Omega\,X)_n$,

$$\alpha_{\Omega\,X}(\mathsf{var}\,x) = x,$$

and for every coherence cell $c = \mathsf{coh}(B, A, \tau)$,

$$\alpha_{\Omega\,X}(\mathsf{coh}(B, A, \tau)) = \alpha_X(\mathsf{coh}(\Sigma\,B, \Sigma^{\mathsf{Sphere}}\,A, \mathsf{Free}\,\kappa \circ \Sigma\,\tau)).$$

In particular, given a Batanin tree B, a full $(n-1)$-sphere A of $\mathsf{Free}\,\mathsf{Pos}(B)$, and a diagram $\tau : \mathsf{Pos}(B) \to \Omega\,X$, to form the composite of this diagram

$$\alpha_{\Omega\,X}(\mathsf{coh}(B, A, \mathsf{Free}\,\tau)) \in (\Omega\,X)_n \subseteq X_{n+1}$$

we view τ as a diagram $\tau^\dagger : \mathsf{Pos}(\Sigma\,B) \to X$ in X of higher dimension, we compose this diagram in X, and then view this as a cell of the hom ω-category $\Omega\,X$:

$$\alpha_{\Omega\,X}(\mathsf{coh}(B, A, \mathsf{Free}\,\tau)) = \alpha_X(\mathsf{coh}(\Sigma\,B, \Sigma^{\mathsf{Sphere}}\,A, \mathsf{Free}\,\tau^\dagger)).$$

5.3 Opposites of hom ω-categories

As a final application of the techniques introduced in this paper, we show that the operations of forming hom ω-categories and opposite categories commute. To make this statement precise, we first extend the action of $G = \mathbb{Z}_2^{\mathbb{N}_{>0}}$ on $\omega\,\mathsf{Cat}$ to an action on bipointed ω-categories

$$\mathsf{op} : G \to \mathsf{Aut}(\omega\,\mathsf{Cat}^{\star,\star})$$

by letting for $w \in G$ the opposite of a bipointed ω-category (X, x, x_+) be the ω-category $\mathsf{op}_w\,X$ with the same basepoints when $1 \notin w$, and with the basepoints swapped when $1 \in w$.

Lemma 13. *For every $w \in G$, there exists a natural isomorphism*

$$\mathsf{op}^{\Sigma}_w : \Sigma\,\mathsf{op}_{w-1} \Rightarrow \mathsf{op}_w\,\Sigma : \mathsf{Comp} \to \mathsf{Comp}^{\star,\star}$$

compatible with the natural isomorphism of Lemma 4 in the sense that

$$\mathsf{op}^{\Sigma}_w\,\mathsf{Free} = \mathsf{Free}\,\mathsf{op}^{\Sigma}_w$$

and the following diagram commutes:

$$\begin{tikzcd}
\Sigma\, \mathsf{op}_{w-1}\, \mathsf{Cell} \arrow[r, Rightarrow, "\mathsf{op}_w^{\Sigma}\, \mathsf{Cell}"] \arrow[d, Rightarrow, "\Sigma\, \mathsf{op}_{w-1}^{\mathsf{Cell}}"'] & \mathsf{op}_w\, \Sigma\, \mathsf{Cell} \arrow[d, Rightarrow, "\mathsf{op}_w\, \Sigma^{\mathsf{Cell}}"] \\
\Sigma\, \mathsf{Cell}\, \mathsf{op}_{w-1} \arrow[d, Rightarrow, "\Sigma^{\mathsf{Cell}}\, \mathsf{op}_{w-1}"'] & \mathsf{op}_w\, \mathsf{Cell}\, \Sigma \arrow[d, Rightarrow, "\mathsf{op}_w^{\mathsf{Cell}}\, \Sigma"] \\
\mathsf{Cell}\, \Sigma\, \mathsf{op}_{w-1} \arrow[r, Rightarrow, "\mathsf{Cell}\, \mathsf{op}_w^{\Sigma}"'] & \mathsf{Cell}\, \mathsf{op}_w\, \Sigma
\end{tikzcd}$$

Proof. We will build natural isomorphism

$$\mathsf{op}_w^{\Sigma} : \Sigma\, \mathsf{op}_{w-1} \Rightarrow \mathsf{op}_w\, \Sigma : \mathsf{Comp}_n \to \mathsf{Comp}_{n+1}$$

inductively on $n \geq -1$ commuting with the forgetful functors u_n, the inclusion functors Free_n and making the following pentagons commute

$$\begin{tikzcd}
& \mathsf{Cell}_n \arrow[dl, Rightarrow, "\mathsf{op}_{w-1}^{\mathsf{Cell}}"'] \arrow[dr, Rightarrow, "\Sigma^{\mathsf{Cell}}"] & \\
\mathsf{Cell}_n\, \mathsf{op}_{w-1} \arrow[d, Rightarrow, "\Sigma^{\mathsf{Cell}}\, \mathsf{op}_{w-1}"'] & & \mathsf{Cell}_{n+1}\, \Sigma \arrow[d, Rightarrow, "\mathsf{op}_w^{\mathsf{Cell}}\, \Sigma"] \\
\mathsf{Cell}_{n+1}\, \Sigma\, \mathsf{op}_{w-1} \arrow[rr, Rightarrow, "\mathsf{Cell}_{n+1}\, \mathsf{op}_w^{\Sigma}"] & & \mathsf{Cell}_{n+1}\, \mathsf{op}_w\, \Sigma
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathsf{Sphere}_n \arrow[dl, Rightarrow, "\mathsf{op}_{w-1}^{\mathsf{Sphere}}"'] \arrow[dr, Rightarrow, "\Sigma^{\mathsf{Sphere}}"] & \\
\mathsf{Sphere}_n\, \mathsf{op}_{w-1} \arrow[d, Rightarrow, "\Sigma^{\mathsf{Sphere}}\, \mathsf{op}_{w-1}"'] & & \mathsf{Sphere}_{n+1}\, \Sigma \arrow[d, Rightarrow, "\mathsf{op}_w^{\mathsf{Sphere}}\, \Sigma"] \\
\mathsf{Sphere}_{n+1}\, \Sigma\, \mathsf{op}_{w-1} \arrow[rr, Rightarrow, "\mathsf{Sphere}_{n+1}\, \mathsf{op}_w^{\Sigma}"] & & \mathsf{Sphere}_{n+1}\, \mathsf{op}_w\, \Sigma
\end{tikzcd}$$

For the unique (-1)-computad, we let

$$\mathsf{op}_w^{\Sigma} : \{v_-, v_+\} \to \{v_-, v_+\}$$

be the identity function when $1 \notin w$, and the function swapping the two generators when $1 \in w$. This is a natural isomorphism making the second pentagon commute: both sides send the unique (-1)-sphere to the 0-sphere (v_-, v_+) when $1 \notin w$, and to the 0-sphere (v_+, v_-) otherwise.

For an n-computad $C = (C_{n-1}, V_n^C, \phi_n^C)$ where $n \in \mathbb{N}$, we let

$$\mathsf{op}_{w,C}^{\Sigma} = (\mathsf{op}_{w,C_{n-1}}^{\Sigma}, \mathsf{var}) : \Sigma\, \mathsf{op}_{w-1}\, C \to \mathsf{op}_w\, \Sigma\, C$$

This is a well-defined morphism of computads by the commutativity of the second pentagon one dimension lower. Moreover, it commutes with the forgetful and the free functors by construction.

We will show that the first pentagon commutes for a computad C and that op_w^{Σ} is natural mutually inductively. First we see that the pentagon commutes when restricted to generators, since both $\mathsf{op}_w^{\mathsf{Cell}}$ and Σ^{Cell} preserve generators. Suppose now that $n > 0$ and let $c = \mathsf{coh}(B, A, \tau)$ a coherence n-cell of C. Then we see that

$$\begin{aligned}
&(\mathsf{op}_{w,\Sigma C}^{\mathsf{Cell}} \circ \Sigma_C^{\mathsf{Cell}})(c) \\
&\qquad = \mathsf{coh}(\mathsf{op}_w \Sigma B, A_1, \mathsf{op}_w \Sigma \tau \circ \mathsf{Free}_n \mathsf{op}_w^{\Sigma B}) \\
&(\mathsf{Cell}_{n+1}(\mathsf{op}_{w,C}^{\Sigma}) \circ \Sigma_{\mathsf{op}_{w-1} C}^{\mathsf{Sphere}} \circ \mathsf{op}_{w-1,C}^{\mathsf{Sphere}})(c) \\
&\qquad = \mathsf{coh}(\Sigma \mathsf{op}_{w-1} B, A_2, \mathsf{op}_{w,C}^{\Sigma} \circ \Sigma \mathsf{op}_{w-1} \tau \circ \Sigma \mathsf{Free}_n \mathsf{op}_{w-1}^{B})
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= \mathsf{Sphere}_n \mathsf{Free}_n (\mathsf{op}_w^{\Sigma B})^{-1} (\mathsf{op}_w^{\mathsf{Sphere}} \Sigma^{\mathsf{Sphere}} A) \\
A_2 &= \Sigma_{\mathsf{op}_{w-1} C}^{\mathsf{Sphere}} (\mathsf{Sphere}_{n-1} \mathsf{Free}_{n-1} (\mathsf{op}_{w-1}^{B})^{-1} (\mathsf{op}_{w-1}^{\mathsf{Sphere}} A))
\end{aligned}$$

By definition of the suspension and the opposite of a tree, we have that

$$\mathsf{op}_w \Sigma B = \Sigma \mathsf{op}_{w-1} B,$$

so the trees over which those coherence cells are built agree. Moreover, the natural isomorphism $\mathsf{op}_w^{\Sigma B}$ is defined to be the composite

$$\mathsf{op}_w^{\Sigma B} = \mathsf{op}_{w,\mathsf{Pos}(B)}^{\Sigma} \circ \Sigma \mathsf{op}_{w-1}^{B}.$$

Using that fact and the naturality of Σ^{Sphere}, we can rewrite the spheres A_1 and A_2 respectively as

$$\begin{aligned}
A_1 &= \mathsf{Sphere}_n \mathsf{Free}_n \Sigma (\mathsf{op}_{w-1}^{B})^{-1} (\mathsf{Sphere}_n \mathsf{Free}_n (\mathsf{op}_{w,\mathsf{Pos}(B)}^{\Sigma})^{-1} (\mathsf{op}_w^{\mathsf{Sphere}} \Sigma^{\mathsf{Sphere}} A)) \\
A_2 &= \mathsf{Sphere}_n \Sigma \mathsf{Free}_{n-1} (\mathsf{op}_{w-1}^{B})^{-1} (\Sigma_{\mathsf{op}_{w-1} C}^{\mathsf{Sphere}} \mathsf{op}_{w-1}^{\mathsf{Sphere}} A)
\end{aligned}$$

and observe that they agree by commutativity of the diagram for spheres one dimension lower, and commutativity of the suspension with the functor Free_n. Moreover, we may assume that the naturality square for τ commutes by the inductive hypothesis, which shows that the morphisms defining the coherence cells agree. Therefore, the first pentagon commutes on coherence cells as well.

Let now $\sigma : D \to C$ be a morphism of n-computads and suppose that the pentagon commutes when restricted to cells of the form $\sigma_{n,V}(v)$ for $v \in V_n^D$. To show that the naturality square for σ commutes, i.e. that

$$\mathsf{op}_{w,C}^{\Sigma} \circ \Sigma \mathsf{op}_{w-1} \sigma = \mathsf{op}_w \Sigma \sigma \circ \mathsf{op}_{w,D}^{\Sigma},$$

we may assume by induction on the dimension and commutativity with the forgetful functors that the underlying morphisms of n-computads agree. It remains to show that the two morphisms agree on top-dimensional generators. Let therefore $v \in V_{n+1}^{\Sigma \mathsf{op}_{w-1} D} = V_n^D$ be a generator. Then

$$\begin{aligned}
(\mathsf{op}^{\Sigma}_{w,C} \circ \Sigma \mathsf{op}_{w-1} \sigma)_V(v) &= \mathsf{Cell}_{n+1}(\mathsf{op}^{\Sigma}_{w,C})(\Sigma^{\mathsf{Cell}}_{\mathsf{op}_{w-1} C} \mathsf{op}^{\mathsf{Cell}}_{w-1,C}(\sigma_V(v))) \\
&= \mathsf{op}^{\mathsf{Cell}}_{w,\Sigma C} \Sigma^{\mathsf{Cell}}_{C}(\sigma_V(v)) \\
&= (\mathsf{op}_w \Sigma \sigma)_V(v) \\
&= \mathsf{Cell}_{n+1}(\mathsf{op}_w \Sigma \sigma)(\mathsf{var}\, v) \\
&= \mathsf{Cell}_{n+1}(\mathsf{op}_w \Sigma \sigma)((\mathsf{op}^{\Sigma}_{w,D})_V v) \\
&= (\mathsf{op}_w \Sigma \sigma \circ \mathsf{op}^{\Sigma}_{w,D})_V(v)
\end{aligned}$$

so the two morphisms agree on generators as well. Hence, the naturality square commutes.

This concludes the induction on $n \in \mathbb{N}$. By commutativity with the forgetful functors, the natural isomorphisms op^{Σ}_w for every $n \in \mathbb{N}$ combine to a natural isomorphism

$$\mathsf{op}^{\Sigma}_w : \Sigma \mathsf{op}_{w-1} \Rightarrow \mathsf{op}_w \Sigma : \mathsf{Comp} \to \mathsf{Comp}$$

as well. Commutativity of the first pentagon shows that the diagram of the lemma commutes, since $\mathsf{op}^{\Sigma}_w \mathsf{Cell}$ is the identity on positive-dimensional cells. $\square$

Proposition 14. *For every $w \in G$, the following diagram commutes*

$$\begin{tikzcd}
\omega\,\mathsf{Cat}^{\star,\star} \arrow[r, "\mathsf{op}_w"] \arrow[d, "\Omega"'] & \omega\,\mathsf{Cat}^{\star,\star} \arrow[d, "\Omega"] \\
\omega\,\mathsf{Cat} \arrow[r, "\mathsf{op}_{w-1}"'] & \omega\,\mathsf{Cat}
\end{tikzcd}$$

Proof. In order to prove commutativity of this diagram for some $w \in G$, it is useful to prove commutativity of the analogous diagram on the level of globular sets first:

$$\begin{tikzcd}
\mathsf{Glob}^{\star,\star} \arrow[r, "\mathsf{op}_w"] \arrow[d, "\Omega"'] & \mathsf{Glob}^{\star,\star} \arrow[d, "\Omega"] \\
\mathsf{Glob} \arrow[r, "\mathsf{op}_{w-1}"'] & \mathsf{Glob}
\end{tikzcd}$$

The mate of the natural isomorphism of Lemma 4 is a natural transformation fitting in this square defined as the whiskered composite

$$\mathsf{op}_{w-1} \Omega = \Omega \Sigma \mathsf{op}_{w-1} \Omega \xRightarrow{\Omega\, \mathsf{op}^{\Sigma}_w \Omega} \Omega \mathsf{op}_w \Sigma \Omega \xRightarrow{\Omega\, \mathsf{op}_w \kappa} \Omega \mathsf{op}_w$$

for κ the counit of the adjunction $\Sigma \dashv \Omega$. One of the snake equations of this adjunction states that $\Omega \kappa$ is an identity. Combining that with the fact that op_w preserves cells and acts trivially on morphisms, we see that $\Omega \mathsf{op}_w \kappa$ must also be an identity. Moreover, the natural isomorphism op^{Σ}_w was defined to be the identity on positive-dimensional cells, so $\Omega\, \mathsf{op}^{\Sigma}_w$ must also be an identity. Since the mate of op_w^{Σ} is an identity natural transformation, we conclude that the square above must commute.

Since the diagram commutes on the level of globular sets, and the forgetful functors U^T and $U^{T^{\star,\star}}$ are faithful, it follows that the diagram commutes on the level of ω-categories if it commutes for objects, meaning that for every bipointed ω-category X, the ω-categories $\Omega\,\mathsf{op}_w\,X$ and $\mathsf{op}_{w-1}\,\Omega\,X$ are equal. Both ω-categories have the same underlying globular set by commutativity of the square on the level of globular sets, so it remains to show that they have the same structure morphisms. Unwrapping the definitions of op_w and Ω, this amounts to the commutativity of the following diagram of natural transformations

$$\begin{tikzcd}
T\,\mathsf{op}_{w-1}\,\Omega \arrow[r, Rightarrow, "\mathsf{op}_{w-1}^{T}\,\Omega"] \arrow[d, equal] & \mathsf{op}_{w-1}\,T\,\Omega \arrow[r, Rightarrow, "\mathsf{op}_{w-1}\,\Omega^{T}"] & \mathsf{op}_{w-1}\,\Omega\,T^{\star,\star} \arrow[d, equal] \\
T\,\Omega\,\mathsf{op}_w \arrow[r, Rightarrow, "\Omega^{T}\,\mathsf{op}_w"'] & \Omega\,T^{\star,\star}\,\mathsf{op}_w \arrow[r, Rightarrow, "\Omega\,\mathsf{op}_w^{T^{\star,\star}}"'] & \Omega\,\mathsf{op}_w\,T^{\star,\star}
\end{tikzcd}$$

where $\mathsf{op}_w^{T^{\star,\star}}$ is simply op_w^{T} seen as a natural isomorphism between bipointed globular sets. By naturality of the mate correspondence, commutativity of this diagram is equivalent to that of the following one:

$$\begin{tikzcd}
\Sigma\,T\,\mathsf{op}_{w-1} \arrow[r, Rightarrow, "\Sigma\,\mathsf{op}_{w-1}^{T}"] \arrow[d, Rightarrow, "\Sigma^{T}\,\mathsf{op}_{w-1}"'] & \Sigma\,\mathsf{op}_{w-1}\,T \arrow[r, Rightarrow, "\mathsf{op}_w^{\Sigma}\,T"] & \mathsf{op}_w\,\Sigma\,T \arrow[d, Rightarrow, "\mathsf{op}_w\,\Sigma^{T}"] \\
T^{\star,\star}\,\Sigma\,\mathsf{op}_{w-1} \arrow[r, Rightarrow, "T^{\star,\star}\,\mathsf{op}_w^{\Sigma}"'] & T^{\star,\star}\,\mathsf{op}_w\,\Sigma \arrow[r, Rightarrow, "\mathsf{op}_w^{T^{\star,\star}}\,\Sigma"'] & \mathsf{op}_w\,T^{\star,\star}\,\Sigma
\end{tikzcd}$$

Replacing each op_w^{T} by each inverse and rotating the diagram, we are left to show that the following diagram commutes:

$$\begin{tikzcd}
\Sigma\,\mathsf{op}_{w-1}\,T \arrow[r, Rightarrow, "\Sigma\,\mathsf{op}_{w-1}^{\mathsf{Cell}}\,\mathsf{Free}"] \arrow[d, Rightarrow, "\mathsf{op}_w^{\Sigma}\,T"'] & \Sigma\,T\,\mathsf{op}_{w-1} \arrow[r, Rightarrow, "\Sigma^{\mathsf{Cell}}\,\mathsf{Free}\,\mathsf{op}_{w-1}"] & T^{\star,\star}\,\Sigma\,\mathsf{op}_{w-1} \arrow[d, Rightarrow, "T^{\star,\star}\,\mathsf{op}_w^{\Sigma}"] \\
\mathsf{op}_w\,\Sigma\,T \arrow[r, Rightarrow, "\mathsf{op}_w\,\Sigma^{\mathsf{Cell}}\,\mathsf{Free}"'] & \mathsf{op}_w\,T^{\star,\star}\,\Sigma \arrow[r, Rightarrow, "\mathsf{op}_w^{\mathsf{Cell}^{\star,\star}}\,\mathsf{Free}^{\star,\star}\,\Sigma"'] & T^{\star,\star}\,\mathsf{op}_w\,\Sigma
\end{tikzcd}$$

But this is precisely the diagram of Lemma 13 whiskered on the right with Free, hence it commutes. □

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